

# Optimization of Mineral Resource Extraction and Capital Allocation by Iterative Dynamic Programming \*

Rein Luus

Department of Chemical Engineering, University of Toronto  
Toronto, ON M5S 3E5, Canada  
e-mail: luus@chem-eng.utoronto.ca

## ABSTRACT

An intertemporal resource extraction and capital, both real and “book value”, allocation problem faced by the mineral mining industry is investigated. The mathematical model consists of 8 differential equations and there are 6 control variables. The time horizon of 12 years is chosen, over which optimization is carried out. The numerous state constraints make this a challenging optimal control problem, but with the use of penalty functions the state constraints are readily handled, and the optimal policy is established by iterative dynamic programming.

**Key Words:** Global optimization; Iterative dynamic programming; Nonrenewable resource; Resource taxation

## 1. Introduction

The mineral mining industry, to be internationally competitive and domestically efficient, must respond to its changing business environment. Its domestic tax environment is changing. Canada’s Department of Finance [4] has offered the industry a tax package whose main elements consist of the following proposed changes (i) the elimination of the corporate capital tax  $\tau_{fc}$ , (ii) a reduction in the corporate income rate  $u_f$ , (iii) the elimination of the resource allowance  $\sigma$ , (iv) the introduction of a deduction for actual Crown royalties and other taxes paid  $\gamma$ , and (v) the introduction of a new mineral exploration tax credit  $\phi$ . The package offers to phase in these five tax instruments over a twelve year time horizon according to Table 1.

With a substantial tax change in its business environment the industry must assess the new tax regime’s effect upon its capital allocation program. An average or representative mineral mining firm (hereafter firm), representing the industry and behaving competitively, thus wishes to solve its intertemporal capital problem given the proposed conditions. The

Table 1: Federal proposed transition package

	2004	2005	2006	2007	2008-16
$\tau_{fc}$	0.002	0.00175	0.00125	0.000625	0.0
$u_f$	0.26	0.25	0.23	0.21	0.0
$\sigma$	0.75	0.65	0.35	0.0	0.0
$\gamma$	0.25	0.35	0.65	1.0	1.0
$\phi$	0.07	0.1	0.1	0.1	0.1

firm is thus faced with an optimal control problem in determining the best resource allocation policy.

The model we use is an extended version of the one in Boadway et al. [2], which pictures the firm to be a three stage producer of mineral ore. In the first stage, the firm explores for or acquires a depletable asset deposit. It obtains its deposit by applying “exploration inputs” (a combination of labour and associated capital assets which are expensed) to its deposit acquisition function. In the second stage, the firm develops (increases the capacity to extract) the deposit by the application of depreciable capital (say shafts, construction as well as machinery and equipment) to the outflow of the depletable resource. In the third or extraction stage, the firm concentrates and smelts the depletable asset into marketable ore.

The firm faces the standard challenge. Choose, subject to constraints, the allocation paths of (i) exploration expenditures, (ii) development capital expenditures, (iii) marketable mineral ore output extraction rate, and (iv) capital consumption allowance so as to maximize shareholder wealth. Since the wealth maximization problem is equivalent to a cash flow maximization problem [1], and since the latter problem is easier to solve, then the goal is for the firm to maximize its real cash flows.

This paper offers two contributions. The first is that iterative dynamic programming (IDP) [8] can provide a solution to a difficult issue in economics, namely, the provision of computed (transition) paths

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Table 2: Federal tax instruments

Name	Symbol	Value
RAfRate	$\sigma_0$	0.25
CCARate	$\alpha$	0.25

Table 3: Provincial tax instruments

Name	Symbol	Value
Prov Corp Rate	$u_p$	0.14
Prov Corp Cap Rate	$\tau_{pc}$	0.003
Prov Min Tax Rate	$\tau_{pm}$	0.12
Prov CCARate	$\alpha_m$	0.30
Prov ProcAllow	$\xi_m$	0.20

to firm’s intertemporal extraction and allocation problems. The paper demonstrates that mining firm’s real cash flow optimal paths for exploration input, capital input, mineral output extraction, and Federal plus Provincial capital consumption allowed rates can be computed to address the proposed tax transition paths. The second contribution, similar in spirit to Wielhouwer et al. [12], relates to the optimality of capital consumption allowance rate. For model firms the optimal CCA rate, at the early part of the time horizon, is less than the allowed maximum. The optimal rate the increases as terminal time approaches, reaching the maximum at the terminal time.

## 2. The Control Challenge

In the following paragraphs the proposed tax changes are incorporated, as parameters, into an optimal control problem formulation for the firm’s real cash flow maximization problem. We start with a description of the tax and economic environment in which the firm finds itself. Federal and Provincial government related parameters, as well the financial markets variables, are given in Tables 2-4. In Table 2, RAfRate denotes the Federal Resource Allowance rate.

Table 4: Financial parameters

Name	Symbol	Value
Debt/Asset	$b$	0.4
Price Growth	$g$	0.0025
Inflation	$\hat{\pi}$	0.02
Ec Dep Rate	$\delta$	0.10
Interest Rate	$i$	0.05
Mine Risk Prem	$h$	0.093
3 Month TB	$i^{TB}$	0.0202
Equity Return	$\rho$	$i^{TB} + h$

The firm’s optimal control problem is formulated in the following manner. The planning horizon is twelve years thus  $T = 12$ . The state of the firm is characterized by an eight component point  $x \in X \subset \mathbb{R}_+^8$ . The state space components are, respectively,  $x_1(t)$ , the intangible exploration input;  $x_2(t)$ , the depletable resource (mineral ore);  $x_3(t)$ , the mine’s processed output (concentrated ore);  $x_4(t)$ , the investment flow of depreciable capital (utilized either at the development stage or at the processing stage);  $x_5(t)$  the federal capital consumption rate which the firm elects to choose, and,  $x_6(t)$ , the provincial capital consumption rate chosen by the firm. State variable,  $x_7(t)$ , represents the book value of federal depreciable capital stock, while state variable,  $x_8(t)$ , represents the book value of depreciable capital stock for provincial corporate income tax liability purpose.

The control space of the firm is denoted by  $\mathcal{V}$ . Conditional on tax parameters the firm chooses its six component control vector  $\mathbf{v}(t) = [v_1(t) \ \cdots \ v_6(t)]^T \in \mathcal{V}$ . The first three control components representing the exploration, development and extraction mining processes are, respectively,  $v_1(t)$  the growth rate of the intangible exploration and development investment input,  $v_2(t)$  the growth rate of the mineral nonrenewable resource ore deposit,  $v_3(t)$  the growth rate of the mine output (the extracted processed and marketed ore). The fourth component  $v_4(t)$ , is the depreciable capital<sup>1</sup> investment flow; such flow is directed to the stage of either deposit development or concentration. The fifth and sixth components relate to the mining firm’s election of the level of the CCA rates. Thus the fifth component  $v_5(t)$  is the growth rate the firm chooses for the federal CCA rate, and the last component  $v_6(t)$  is the growth rate the firm chooses for the provincial CCA rate.

The annual inflation rate is two percent thus  $\hat{\pi} = 0.02$ . The firm possesses perfect myopic foresight, thus expectations and realizations coincide. Industry revenues, costs, assets, quantities plus implicit prices and quantities are bench-marked on Ontario Mining Association data [11]. The implicit price of mine output marketable metal ore is a value weighted average of the four major metal ore prices, consisting of the price of copper, nickel, gold and zinc. The real price of marketable processed ore is  $p(t) = 0.4048e^{gt}$ . The real price of depletable capital is given by  $q(t) = 1$ . The real price of the intangible exploration input price is  $w(t) = 0.1768$ .

The exploration and acquisition function<sup>2</sup> for the depletable resource is  $f^a(x_1) = (adjl)x_1$  with parameter value  $adjl = 51.71$ . The development function,

<sup>1</sup>Capital, for the purpose of the model, is malleable.

<sup>2</sup>There are many exploration processes characterized in the natural resources literature. For example, one which integrates heterogeneous mineral deposits and a Bayesian learning process is the one by Cairns and Van Quyen [3].

denoted by  $f^d(x_2, x_4)$  describes a deposit's production capacity of depletable nonrenewable resource. The development function applies depreciable capital  $x_4(t)$  to depletable resource  $x_2(t)$  to produce (develop) more depletable resource from the deposit in question and it is given by  $f^d(x_2, x_4) = (adjx)x_2^{(1-\Gamma)}x_4^\Gamma$  with parameter values  $adjx = 1.141$  and  $\Gamma = 0.116$ . The cost function  $C(x_3, x_4)$ , characterizes the cost of extracting<sup>3</sup> and processing the  $x_3(t)$  flow (units per time period) of processed mineral ore as mine output. The flow of output is produced with the aid of associated depreciable capital  $x_4(t)$  which, in turn, is used in the extraction, concentration and smelting process stages. The cost function, relating to the development and extraction stages, is given by  $C(x_3, x_4) = (adjc)x_3^{1/h1}x_4^{-\Lambda}$  with parameter values  $adjc = 1.11$  and  $h1 = 0.884$  and  $\Lambda = 0.256$ . To finance real capital investment expenditures, the cost of finance capital  $R$  is determined by financial markets and that cost is given by

$$R = bi[1 - u_f(1 + \gamma\tau_{pm}\xi_m) - u_p] + (1 - b)\rho, \quad (1)$$

in which  $b$  is the (industry) debt to assets ratio  $i$  the long term bond interest rate, while  $\rho$  is the risk adjusted return to equity.

The firm's decision challenge is to find the control  $\mathbf{v} \in \mathcal{V} = [-1, +1]^3 \times \mathfrak{R} \times [-1, +1]^2$  strategy which maximizes the performance index

$$\begin{aligned} I[\mathbf{x}, \mathbf{v}] = & \int_0^T e^{-(R-\hat{\pi})t} \left\{ [1 - u_f(1 - \sigma\sigma_0 - \gamma\tau_{pm}(1 - \xi_m)) \right. \\ & - u_p - \tau_{pm}(1 - \xi_m)](px_3 - (adjc)x_3^{1/h1}x_4^{-\Lambda}e^{-\hat{\pi}t}) \\ & - [1 - \phi - u_f(1 - \gamma\tau_{pm}(1 - \xi)) - u_p - \tau_{pm}(1 - \xi_m)]ux_1 \\ & - q(v_4 + \delta x_4) - [\tau_{fc} + \tau_{pc}(1 - u_f)]qx_4 \\ & + [u_f(1 - \sigma\sigma_0) + u_p]x_5x_7e^{-\hat{\pi}t} \\ & \left. - (u_f\gamma - 1)\tau_{pm}(1 - \xi_m)x_6x_8e^{-\hat{\pi}t} \right\} dt, \quad (2) \end{aligned}$$

subject to the equations and constraints related to the operation of the firm.

### 3. Optimal Control Problem Formulation

The integrand in Eq. (2) is a complicated expression, so for simplification of notation we write the integrand as  $\Phi$ . Thus Eq. (2) becomes

$$I[\mathbf{x}, \mathbf{v}] = \int_0^T \Phi dt \quad (3)$$

<sup>3</sup>In the natural resources literature, there are many extraction cost characterizations. For an analysis of costs related to multiple natural resource deposits and the order of extraction see Holland [5].

The firm must satisfy the following constraints:

$$\frac{dx_i(t)}{dt} = v_i(t)x_i(t), \quad \forall i = 1, 2, 3, 5, 6, \quad (4)$$

$$\frac{dx_4(t)}{dt} = -\delta x_4(t) + (v_4(t) + \delta x_4(t))e^{-\hat{\pi}t}, \quad (5)$$

$$\frac{dx_7(t)}{dt} = -x_5(t)x_7(t) + (v_4(t) + \delta x_4(t))qe^{\hat{\pi}t}, \quad (6)$$

$$\frac{dx_8(t)}{dt} = -x_6(t)x_8(t) + (v_4(t) + \delta x_4(t))qe^{\hat{\pi}t}, \quad (7)$$

$$\int_0^T [x_2(t) - adjl.x_1(t)] dt = 0, \quad (8)$$

$$\int_0^T [x_3(t) - adjx.x_2^{1-\Gamma}(t)x_4^\Gamma(t)] dt = 0, \quad (9)$$

$$0 \leq x_5(t) \leq 0.25, \quad (10)$$

$$0 \leq x_6(t) \leq 0.30. \quad (11)$$

The initial state vector is given as

$$\begin{aligned} \mathbf{x}(0) = & [676 \quad 32,952 \quad 9,788 \quad 11,212 \\ & 0.25 \quad 0.30 \quad 11,212 \quad 6,585]^T. \quad (12) \end{aligned}$$

and the final time  $T$  is specified as 12 years. The state variables must be positive semi-definite; the fourth control variable  $v_4$  is unbounded, but the rest of the control variables are bounded by  $-1 \leq v_i \leq 1$ .

There are 8 state variables and 6 control variables, so from the dimensional point of view the problem should not be difficult. The nature of the differential equations for  $x_1, x_2, x_3, x_5,$  and  $x_6,$  keeps these variables positive semi-definite. However,  $x_4, x_7,$  and  $x_8$  require special attention. However, the state constraints expressed by Eq. (10) and Eq. (11), and the final state constraints in Eq. (8) and Eq. (9) also require special attention. To set up the problem for IDP, we first introduce a new state variable  $x_9$  through the equation

$$\frac{dx_9}{dt} = \Phi, \quad x_9(0) = 0, \quad (13)$$

so the performance index to be maximized becomes

$$I = x_9(T). \quad (14)$$

To handle state inequality constraints, as suggested by Mekarapiruk and Luus [10], we introduce auxiliary variables which become positive if the inequality constraints are violated.

First we introduce  $x_{10}, x_{11},$  and  $x_{12}$  to keep  $x_4, x_7,$  and  $x_8$  from becoming negative, by first defining the derivatives of these additional variables to be zero and then

$$\frac{dx_{10}}{dt} = -x_4 \quad \text{if} \quad x_4 < 0, \quad x_{10}(0) = 0, \quad (15)$$

$$\frac{dx_{11}}{dt} = -x_7 \quad \text{if} \quad x_7 < 0, \quad x_{11}(0) = 0, \quad (16)$$

$$\frac{dx_{12}}{dt} = -x_8 \quad \text{if} \quad x_8 < 0, \quad x_{12}(0) = 0, \quad (17)$$

Similarly we introduce  $x_{13}$  and  $x_{14}$ :

$$\frac{dx_{13}}{dt} = x_5 - 0.25 \quad \text{if} \quad x_5 - 0.25 > 0, \quad x_{13}(0) = 0, \quad (18)$$

$$\frac{dx_{14}}{dt} = x_6 - 0.30 \quad \text{if} \quad x_6 - 0.30 > 0, \quad x_{14}(0) = 0. \quad (19)$$

It is clear that if there are no state violations then  $x_{10} - x_{14}$  will be zero at the final time  $T$ .

To handle the equality constraints in Eq. (8) and Eq. (9), we introduce the differential equations

$$\frac{dx_{15}}{dt} = x_2(t) - \text{adjl}.x_1(t), \quad x_{15}(0) = 0, \quad (20)$$

$$\frac{dx_{16}}{dt} = x_3(t) - \text{adj}x_2^{1-\Gamma}(t)x_4^\Gamma(t), \quad x_{16}(0) = 0. \quad (21)$$

Now we introduce the augmented performance index to be maximized

$$J = x_9(T) - \theta_1(x_{15}(T) - s_1)^2 - \theta_2(x_{16}(T) - s_2)^2 - \theta_3x_{10}(T) - \theta_4x_{11}(T) - \theta_5x_{12}(T) - \theta_6x_{13}(T) - \theta_7x_{14}(T). \quad (22)$$

In this augmented performance index we have introduced seven positive penalty function factors  $\theta_i$ ,  $i = 1, \dots, 5$  and two shifting terms  $s_1$  and  $s_2$ . The shifting terms are used to ensure that the equality constraints are satisfied. As was has been shown recently [7],[8] quadratic penalty function by itself is generally inadequate means of ensuring that equality constraints will be satisfied at the optimum even if very large penalty function factors are used. After a few trial runs, it was found that a suitable set of penalty function factors is  $\theta = [1 \ 1 \ 1 \ 10^6 \ 10^6 \ 10^8 \ 10^8]^T$ , where we put a very heavy penalty if the upper constraints of  $x_5$  and  $x_6$  are violated, or if  $x_7$  or  $x_8$  should become negative. The shifting terms  $s_1$  and  $s_2$  are put initially to zero and then are updated after every pass of IDP according to the deviation of the equality constraint as outlined in [8].

#### 4. Iterative Dynamic Programming Algorithm

Details for IDP can be found in [8]. Here, however, we wish to show how the IDP was applied to this specific problem. The following steps were used:

1. Divide the time interval  $[0, T]$  into  $P$  time stages, each of length  $L$ . At each time stage, we seek

a constant value for the control vector  $\mathbf{v}$ . Here we have  $T = 12$ , so we chose  $P$  to be 48, so that each time stage corresponds to a quarter of a year. With 6 control variables, there are thus 288 values to determine. With IDP we seek 6 values 48 times, rather than attempting to get all 288 values simultaneously.

2. Choose the number of test values for the control vector  $\mathbf{v}$  denoted by  $R$ , an initial control policy and the initial region size  $\mathbf{r}_{in}$ ; also choose the region contraction factor  $\gamma$  used after every iteration and the number of grid points  $N$  at each time stage. Here we chose  $R = 200$ , initial control policy  $\mathbf{v} = \mathbf{0}$ ,  $\gamma = 0.95$ ,  $N = 1$ , and allow the initial region size for each control variable to be chosen between 0.001 to 1.0. The use of a different region sizes allows us to check for convergence.

3. Choose the total number of iterations to be used in every pass and set the iteration number index to  $j = 1$ . We used 20 iterations in each pass.

4. Set the region size vector  $\mathbf{r}^j = \mathbf{r}_{in}$ .

5. By using the best control policy (the initial control policy for the first iteration) integrate the equations, from  $t = 0$  to  $T$ ,  $N$  times with different values for control. This will generate  $N$   $\mathbf{x}$ -trajectories which provide the grid points. Store the values of  $\mathbf{x}$  at the beginning of each time stage, so that  $\mathbf{x}(k - 1)$  corresponds to the value of  $\mathbf{x}$  at beginning of stage  $k$ .

6. Starting at stage  $P$ , corresponding to time  $T - L$ , for each of the  $N$  stored values for  $\mathbf{x}(P - 1)$  from step 5 (grid points) integrate the differential equations from  $T - L$  to  $T$ , with each of the  $R$  allowable values for the control vector calculated from

$$\mathbf{v}(P - 1) = \mathbf{v}^{*j}(P - 1) + \mathbf{D}\mathbf{r}^j \quad (23)$$

where  $\mathbf{v}^{*j}(P - 1)$  is the best value obtained in the previous iteration and  $\mathbf{D}$  is a diagonal matrix of different random numbers between  $-1$  and  $1$ . Out of the  $R$  values for the augmented performance index, choose the control values that give the maximum value, and store these values as  $\mathbf{v}(P - 1)$ . We now have the best control for each of these  $N$  grid points.

7. Step back to stage  $P - 1$ , corresponding to time  $T - 2L$ , and for each of the  $N$  grid points do the following calculations. Choose  $R$  values for  $\mathbf{v}(P - 2)$  as in the previous step, and by taking as the initial state  $\mathbf{x}(P - 2)$  integrate the differential equations over one stage length. Continue integration over the last time stage by using the stored value of  $\mathbf{v}(P - 1)$  from step 6 corresponding to the grid point that is closest to the value of the state vector that has been reached. Compare the  $R$  values of the performance index and store the  $\mathbf{v}(P - 2)$  that gives the maximum value for the augmented performance index.

8. Continue the procedure until stage 1, corresponding to the initial time  $t = 0$  and the given initial state, is reached. This stage has only a single grid

point, since the initial state is specified. As before, integrate the differential equations and compare the  $R$  values of the augmented performance index and store the control  $\mathbf{v}(0)$  that gives the maximum augmented performance index. Store also the corresponding  $\mathbf{x}$ -trajectory. This completes one iteration.

9. Reduce the region for allowable control

$$\mathbf{r}^{j+1} = \gamma \mathbf{r}^j \quad (24)$$

where  $j$  is the iteration number index. Use the best control policy from step 8 as the midpoint for the allowable values for the control denoted by the superscript  $*$ .

10. Increment the iteration index  $j$  by 1 and go to step 5 and continue the procedure for the specified number of iterations. This concludes one pass.

11. For the next pass, update the shifting terms according to the deviations

$$s_1^{q+1} = s_1^q - x_{15}(T) \quad (25)$$

$$s_2^{q+1} = s_2^q - x_{16}(T) \quad (26)$$

where  $q$  is the pass number. Now choose a new region size for each of the control variables. Here we chose the new initial region sizes to be the amount by which a control variable changed during the pass. If this was less than  $\epsilon$  then we put the region size equal to  $\epsilon$ . We took  $\epsilon = 10^{-6}$ . In the present work we used 200 passes after which we interpreted the results. Of special interest is to ensure that all the constraints are satisfied.

## 5. Computational Results

The computations were done in double precision using WATCOM FORTRAN compiler version 9.5 on an Intel Pentium4/2.4GHz and an AMD Athlon/XP2000+ personal computers. For integration of the equations DVERK subroutine was used with local error tolerance of  $10^{-5}$  to give reliable results.

We found that the convergence rate was very slow. To get convergence, we used several runs, using the control policy obtained from the previous run as a starting point for the next run. Since no constraints are put on  $v_4$ , for numerical purposes, we initially used the constraint  $-10000 \leq v_4 \leq 10000$ . Convergence to  $I = 6227$  was obtained after several runs. It was found that  $v_4$  was close to zero for the first 47 time stages. Then at the 47<sup>th</sup> stage  $v_4$  switched to  $-7493$ , and at the 48<sup>th</sup> time stage,  $v_4 = -10000$ . All constraints were satisfied. As is shown in Figure 1, the constraints on  $x_5$  and  $x_6$  are satisfied. The trajectories of the state variables  $x_1, x_2, x_3, x_4, x_7$  and  $x_8$  are given in Figure 2. It is interesting to note that the values of the state variables drop off very rapidly in the last two time stages, and  $x_8$  becomes zero at the final time. The control policy is shown in Figures 3-5.

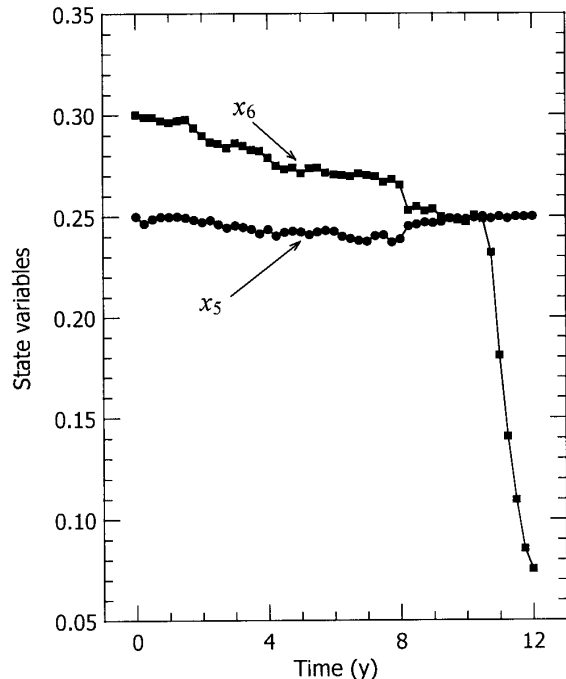


Figure 1: Trajectories of  $x_5$  and  $x_6$  for  $I = 6227$  with  $-10000 \leq v_4 \leq 10000$ , showing that the constraints are not violated for these state variables

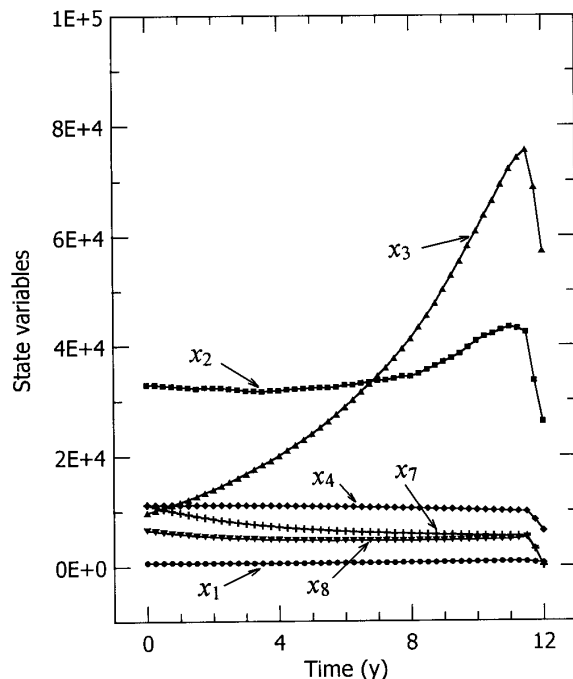


Figure 2: Trajectories of  $x_1, x_2, x_3, x_4, x_7$ , and  $x_8$  for  $I = 6227$ , where  $-10000 \leq v_4 \leq 10000$

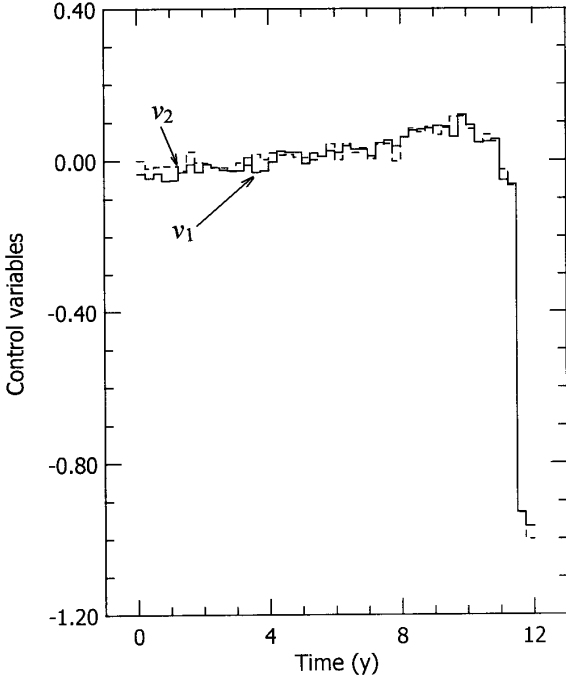


Figure 3: Control variables  $v_1$  and  $v_2$  for  $I = 6227$  with  $-10000 \leq v_4 \leq 10000$

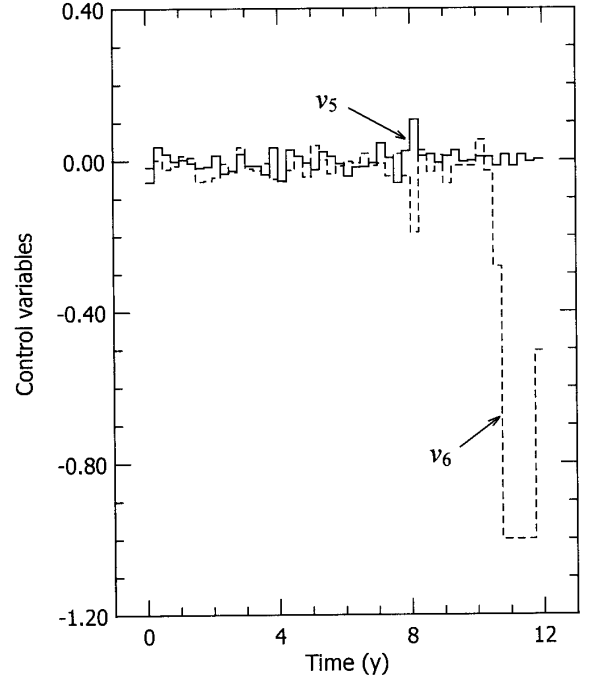


Figure 5: Control variables  $v_5$  and  $v_6$  for  $I = 6227$  with  $-10000 \leq v_4 \leq 10000$

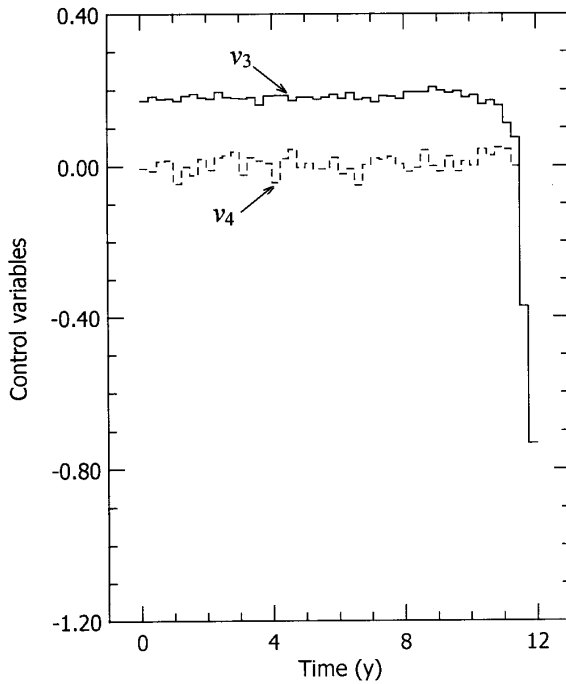


Figure 4: Control variables  $v_3$  and  $v_4$  for  $I = 6227$  with  $-10000 \leq v_4 \leq 10000$ , where  $v_4$  is  $-7493$  and  $-10000$  at the last two stages (not shown)

We now relaxed the bounds on  $v_4$  totally and repeated the runs, using  $R = 200$  randomly chosen values for each iteration, 20 iterations per pass and used

200 passes. The performance index is now increased to  $I = 6248$ . Although the state variables  $x_5$  and  $x_6$  behave similarly to the former case, as is shown in Figure 6, the trajectories for the other state variables are somewhat different during the last two stages, as is shown in Figure 7. Due to the relaxation of the bounds on  $v_4$ , the action of using a very large negative value for  $v_4$  is delayed until the last stage. At that stage  $v_4 = -17100$ .

Although the performance index is now increased only marginally from 6227 to 6248, the control policies are quite different. This illustrates the low sensitivity of the control policy on the performance index and shows why the establishment of the optimal control policy is quite difficult for these types of problems.

#### *Effect of the time horizon*

It is well known that the final time chosen for optimal control, i.e.  $T$ , is an important variable [6]. To examine the effect of the choice of the time horizon, we chose  $T = 6$  years. It was interesting to note that the performance index obtained,  $I = 6367$ , is somewhat larger than that obtained with  $T = 12$ . The state trajectories, as shown in Figures 11 and 12, are totally different. Both  $x_7$  and  $x_8$  become zero at the final time.

Observing the results we obtained with  $T = 6$ , we reran the case with  $T = 12$ , using 7 grid points for the state. The value of the performance index obtained

now was  $I = 6376$ . The state trajectories for  $x_5$  and  $x_6$  are now quite different as can be seen from Figure 13. However, the trajectories for the other state variables are quite similar as is seen in Figure 14. The values of  $v_4$  for the last two time stages are  $-4.471 \times 10^3$  and  $-1.509 \times 10^4$ , respectively. Due to the length of the computation time required with the use of a larger number of grid points, the convergence properties of this system were not investigated in greater detail.

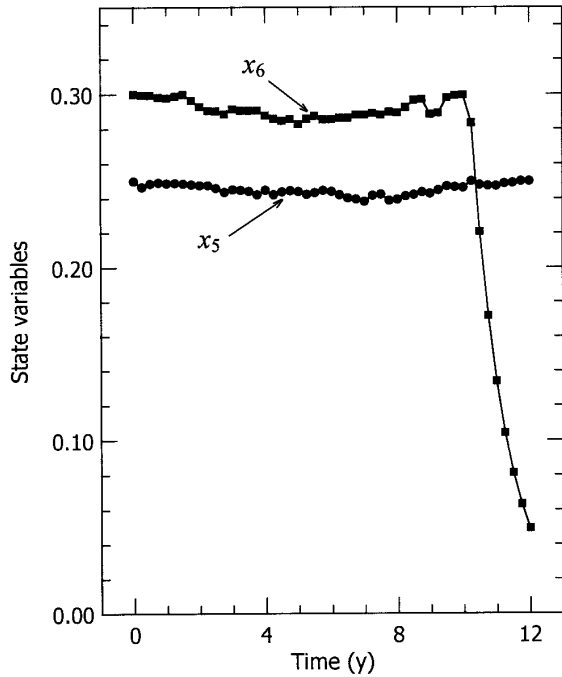


Figure 6: Trajectories of  $x_5$  and  $x_6$  for  $I = 6248$  with unconstrained  $v_4$

## 6. Conclusions

We have shown that IDP can be used to provide solution paths to difficult economic and tax transition issues. Although introduction of accessory variables was necessary to handle state constraints, these accessory variables are well-behaved and are easy to handle. Programming of the problem was straightforward. The convergence rate, however, was very slow, and it is very difficult to get the global optimum for this problem. There are many local optima, and several runs were required to establish the optimal control policy.

Analysis of the state variables showed that under the Department of Finance's proposed tax transition scheme, to maximize their cash flow, the firms should

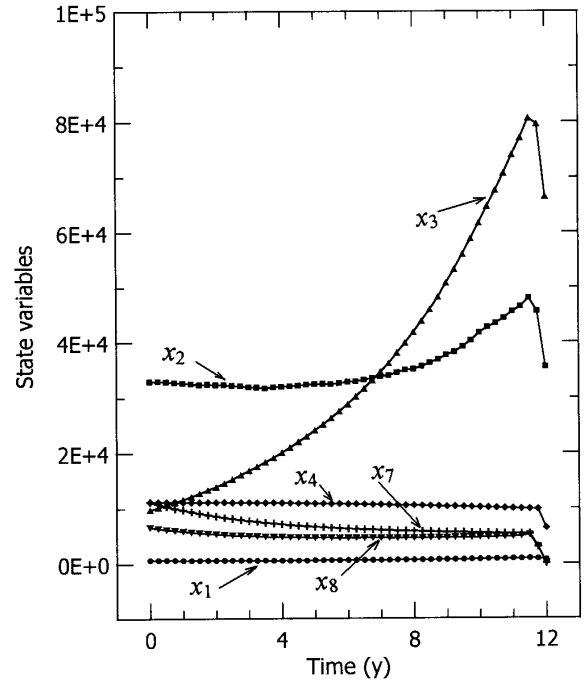


Figure 7: Trajectories of  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ ,  $x_7$ , and  $x_8$  for  $I = 6248$ , where no constraints are placed on  $v_4$

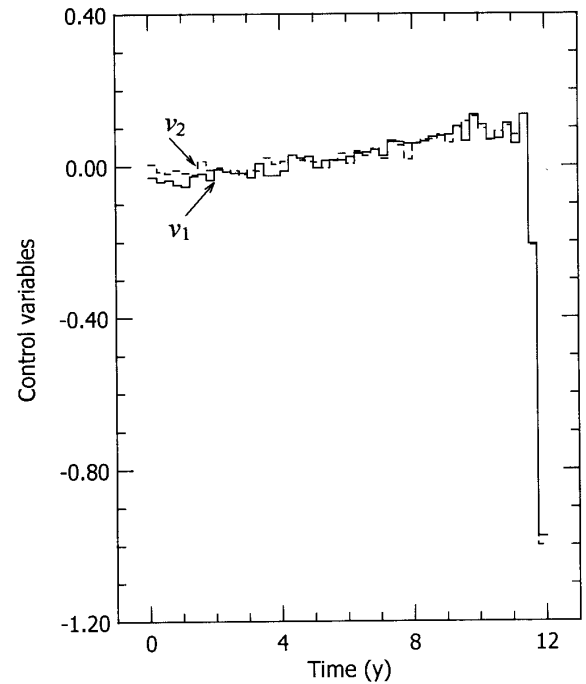


Figure 8: Control variables  $v_1$  and  $v_2$  for  $I = 6248$  with no constraints placed on  $v_4$

extract mineral output at increasing rates, thus extracting most of the mineral resource later rather than earlier. Furthermore, the optimal Federal and Provincial capital consumption allowance rates are not at

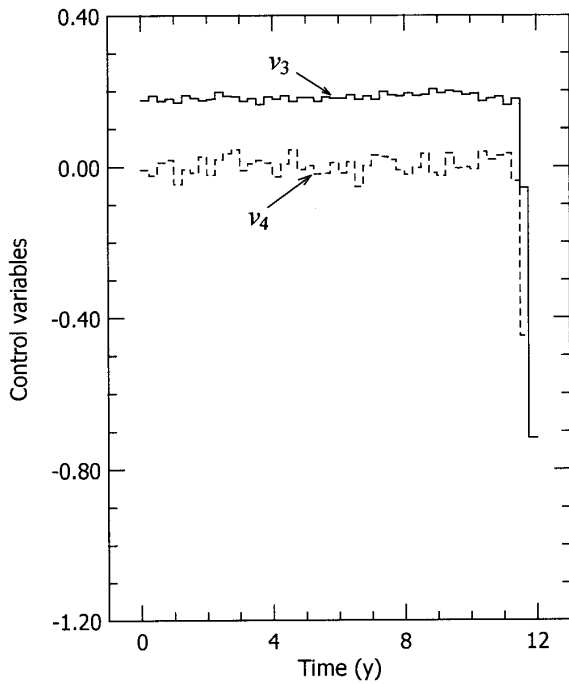


Figure 9: Control variables  $v_3$  and  $v_4$  for  $I = 6248$  with no constraints placed on  $v_4$ , where  $v_4 = -17100$  at the last stage (not shown)

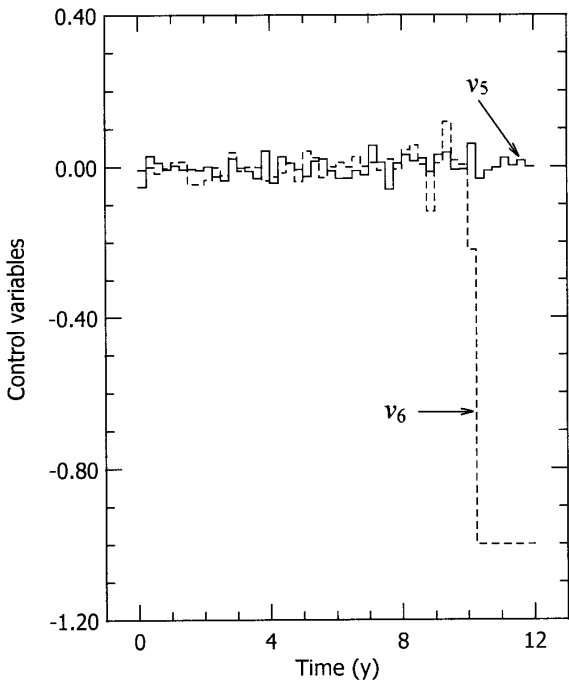


Figure 10: Control variables  $v_5$  and  $v_6$  for  $I = 6248$  with no constraints placed on  $v_4$

their maximum legal bound.

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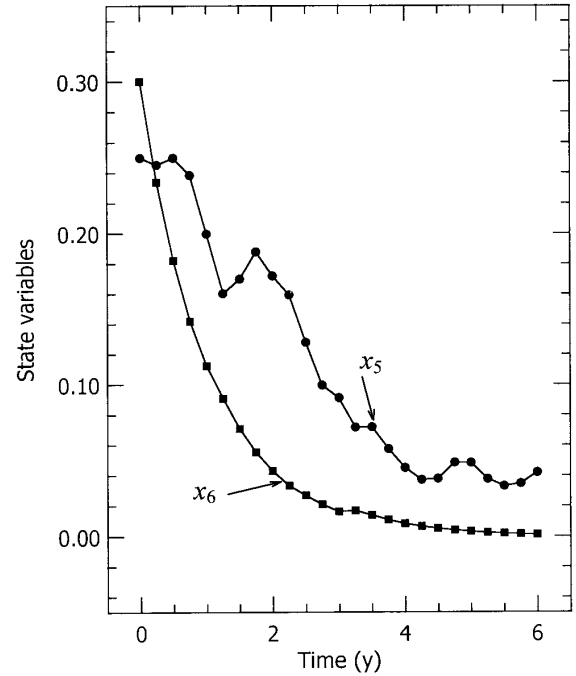


Figure 11: Trajectories of  $x_5$  and  $x_6$  for  $T = 6$ , yielding  $I = 6367$

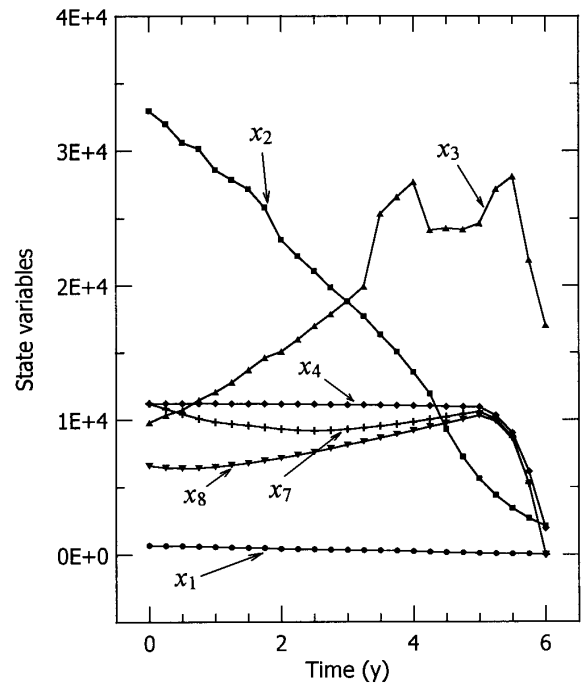


Figure 12: Trajectories of  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ ,  $x_7$ , and  $x_8$  for  $T = 6$ , yielding  $I = 6367$

ternational Business, University of Toronto, for setting up the mathematical model which was used in this study. Financial support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

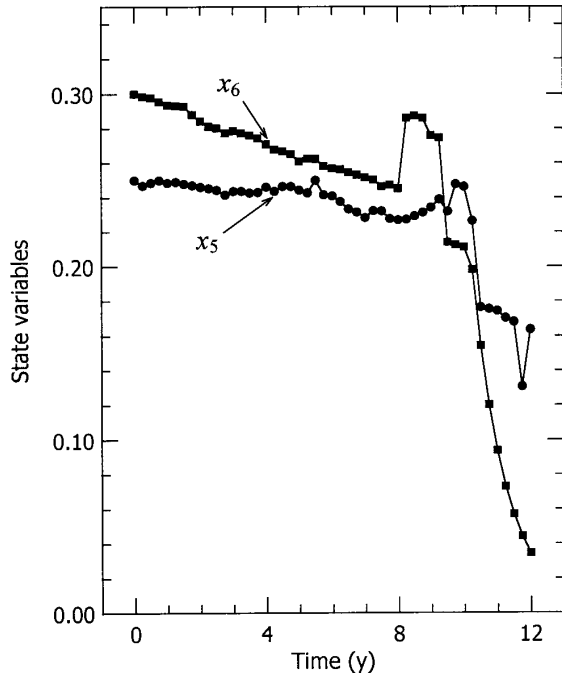


Figure 13: Trajectories of  $x_5$  and  $x_6$  for  $T = 12$ , yielding  $I = 6376$

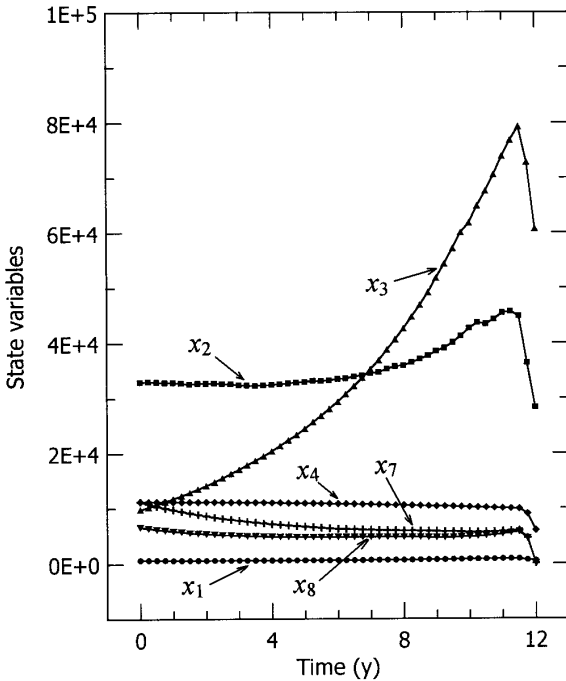


Figure 14: Trajectories of  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ ,  $x_7$ , and  $x_8$  for  $T = 12$ , yielding  $I = 6376$

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