

Sustainable growth: The extraction-saving relationship*

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Abstract

The paper presents two new results for the Dasgupta-Heal-Solow-Stiglitz model with an essential nonrenewable resource:

(1) the pattern of the resource extraction can be more important for the sustainable growth than the pattern of saving when the Hotelling Rule modifier is not small enough;

(2) the qualitative behavior of the long-run per capita consumption can be examined along any smooth enough path of extraction using the “sustainability functional” introduced in the paper.

JEL Classification: O13; O47; Q32; Q38

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1 Introduction

Dasgupta and Heal (1979, pp. 303-306) and Hamilton et al. (2006) showed that the investments exceeding the standard Hartwick Investment Rule (Hartwick, 1977) imply sustainable unbounded growth in per capita consumption under the standard Hotelling Rule for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974).

Stollery (1998) considered an externality (global warming) that implied modification of the Hotelling Rule and corresponding modification of the path of extraction. Combination of this extraction path with the standard Hartwick Rule resulted in sustainable bounded growth of per capita consumption in the long run. Another example was obtained in (Bazhanov, 2007b and 2008a) where I examined the properties of the transition paths constructed under the assumption of the modified Hotelling Rule. This case also gave the patterns of bounded and unbounded growth of per capita consumption under the standard Hartwick Rule.

These examples raise a question about the roles of the patterns of saving and extraction for sustaining the growth of an economy in the long run. The answer on this question is the first main result of the paper (Proposition 1, Section 3). It shows that the pattern of the resource extraction is more important for the sustainability of growth than the pattern of saving when the Hotelling Rule modifier is not close enough to zero. The pattern of saving defines the level of consumption along the growing or declining path in this case. This result is obtained for the DHSS model with the constant-share-of-output investment rule and the modified Hotelling Rule.

The second result (Proposition 2, Section 4) allows for estimation of the sustainability of the resource extraction along any smooth enough path in a

sense of nondecreasing per capita consumption in the long run. The importance of this result is connected with the long-standing problems of defining and estimating the properties of the “possible” and the “optimal” paths of extraction. For example, Dasgupta and Heal (1979) showed that the optimal with respect to the discounted utilitarian criterion path of extraction is unsustainable (consumption declines to zero in the long run). Applying the “sustainability functional” introduced in Proposition 2, I show that the long-run consumption along the well-known Hubbert curve of oil extraction declines to zero regardless of the choice of the curve’s parameters and regardless of the pattern of saving. Numerical examples considered in the paper are calibrated on the current world oil extraction data.

2 The model

I consider the DHSS model with the technical progress exactly compensating for the capital depreciation¹ (Bazhanov, 2007a), zero population growth, and zero extraction cost with the Cobb-Douglas production function $q(t) = f(k(t), r(t)) = k^\alpha(t)r^\beta(t)$ where $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$, are constants. Population equals to labor and the lower-case variables are in per capita units, q – output, k – produced capital, r - current resource use. Then $\dot{r} = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$). Prices of per capita capital and the resource are $f_k = \alpha q/k$, $f_r = \beta q/r$ where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$. I assume that

- (1) the Hotelling Rule is modified by some phenomena in the following

¹This assumption allows for correct consideration of the basic DHSS model in cases with bounded and unbounded growth of consumption.

way

$$\frac{\dot{f}_r(t)}{f_r(t)} = f_k(t) + \tau(t) \quad (1)$$

where the modifier $\tau(t)$ can be identically equal to zero. This allows for the various feasible² scenarios of the resource extraction $r_\tau(t)$. The assumption is essential for the goals of the paper because it implies that all the examined paths of extraction are realizable. Realization of the specific extraction path depends on the concrete paths of the phenomena modifying the Hotelling Rule including the government's policies. A review of these phenomena can be found in (Gaudet, 2007). I imply in this assumption that the government can use all the instruments of influence on the externalities and on the corresponding paths of extraction. For example, the government can use taxes (e.g. Karp and Livernois, 1992), regulations (e.g. Davis and Cairns, 1999), and education (e.g. Grimaud and Rouge, 2005), in order to realize the socially optimal program, using as a feedback the observable path of extraction or paths of the resource prices and the interest rate. The plausibility of the assumption about the possibility to realize the considered paths of extraction is supported by a large body of empirical research in the "oil peak" theory (e.g. Laherrere, 2000).³ For simplicity of notation, I will omit below the subindex τ which denotes the dependence of the path of extraction on the specific combination of the phenomena modifying the standard Hotelling Rule.

²It is assumed here that the path of extraction $r(t)$ is feasible if

- (a) $r(t) > 0$ for all $t \geq 0$;
- (b) the reserve s_0 is extracted during the infinite period $s_0 = \int_0^\infty r(t)dt$;
- (c) the path $r(t)$ is consistent with the initial conditions $r(0) = r_0$; $\dot{r}(0) = \dot{r}_0$;
- (d) $r(t)$ is smooth enough ($\dot{r}(t)$ is continuous for t big enough).

³The main goal of this theory is to explain the behavior of historical data, and then to use these explanations as a forecasting tool. The paths of consumption along some of the curves, being used for these explanations, are examined in this paper.

(2) The economy follows an investment rule in a form of $\dot{k} = wq$ where $w \in (0, 1)$ is a constant. This rule includes the Hartwick Investment Rule as a specific case for $w = \beta$.

3 The roles of extraction and saving for growth

The modified Hotelling Rule (1) implies $\dot{f}_r/f_r = \alpha\dot{k}/k + (\beta-1)\dot{r}/r = \alpha q/k + \tau$ which after substitution of the saving rule $\dot{k} = wq$ becomes $\alpha qw/k + (\beta - 1)\dot{r}/r = \alpha q/k + \tau$. This gives us the generalized equation in $r(t)$

$$\frac{\dot{r}}{r} = -\frac{1-w}{1-\beta} \frac{\alpha q}{k} + \frac{\tau}{\beta-1} \quad (2)$$

This equation can be rewritten as $\dot{r}/r = -[(1-w)f_k + \tau]/(1-\beta)$ or using (1) as

$$\dot{r}/r = -\left[\dot{f}_r/f_r - wf_k\right]/(1-\beta) \quad (3)$$

Note that the last expression does not contain the modifier $\tau(t)$ explicitly because according to (1) the distortions caused by the phenomena associated with $\tau(t)$ are included in the price changes \dot{f}_r/f_r . It means that if we assume that $\tau(t)$ includes all known and unknown effects distorting the Hotelling Rule in the real economy then we must take for numerical examples the changes of the real market price for the term \dot{f}_r/f_r . Then, formula (3) is interpreted as follows: *“actual” rates of extraction are defined by the rates of the “actual” price changes \dot{f}_r/f_r diminished by the interest rate f_k weighed by the investment coefficient w .*⁴

⁴Since all variables in formula (3) are observable, it can be used for the estimation of accuracy of the model for the real economy. I use the word “actual” in quotation marks because the aggregate model obviously reflects only qualitative behavior of the economy with some level of inaccuracy in numbers.

Substitution of formula (2) into the expression for the output per cent change implies $\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r = \alpha q/k [(w - \beta)/(1 - \beta)] - \tau\beta/(1 - \beta)$ which can be rewritten as follows

$$\dot{q}(1 - \beta)/(\beta q) = f_k(w/\beta - 1) - \tau.$$

This implies the following

Proposition 1 *In the DHSS economy with the investment rule $\dot{k} = wq$ where $w \in (0, 1)$ and with the modified Hotelling Rule $\dot{f}_r/f_r = f_k + \tau(t)$, the sign of the change in per capita output (and consumption) is defined as follows*

$$\dot{q} \gtrless 0 \text{ iff } \tau \gtrless f_k [w/\beta - 1]$$

which means that the path of investment (defined by w) can qualitatively influence the pattern of growth only when $-f_k < \tau < f_k [1/\beta - 1]$.

The specific cases of this result are:

1. The necessity and sufficiency of the Hartwick Rule ($w = \beta$) for sustaining the constant per capita consumption in the DHSS model with the standard ($\tau(t) \equiv 0$) Hotelling Rule (Hartwick, 1977; Dixit et al, 1980).
2. The necessity and sufficiency of the modified Hartwick Rule ($w > \beta$) for sustaining per capita growth of consumption in the DHSS model with the standard Hotelling Rule ($\tau(t) \equiv 0$) (Dasgupta and Heal, 1979, p. 303-306, formula (10.33); Hamilton et al, 2006).
3. The growing per capita consumption in the DHSS economy with the modified Hotelling Rule ($\tau(t) < 0$) and the standard Hartwick Rule ($w = \beta$) (Stollery, 1998; Bazhanov, 2007b, 2008a, 2008b).

Note that these results were obtained under the different welfare criteria (maximin in cases 1 and 3, and utilitarian with zero discounting of utility and the specific social rates of time preference in (Dasgupta and Heal, 1979)).

Now I again will substitute for $\tau(t)$ using (1) in order to express the result of Proposition 1 in terms of the rates of change of the observable market prices \dot{f}_r/f_r for our resource. The result of this substitution I will formulate as

Corollary 1. *Under the conditions of Proposition 1 the sign of the change in per capita output (and consumption) is defined as follows*

$$\dot{q} \gtrless 0 \text{ iff } \left[\dot{f}_r/f_r \right] / f_k \lesseqgtr w/\beta$$

which implies that the pattern of saving (w) can qualitatively influence the pattern of growth iff $0 < \left[\dot{f}_r/f_r \right] / f_k < 1/\beta$.

One of the practical implications of this result is that the DHSS model, given the resource input as oil, has some empirical support from the qualitative behavior of the world's economy depending on the major changes in the (market) oil prices. One can recall the prosperity of the economy when the price of oil was declining ($\dot{f}_r/f_r < 0$) or the rate of change was very small before the spike in 1973 and the corresponding recesses after the sharp price spikes in 1973 and 1979.⁵

The examples from the world's history and a large body of empirical research on testing the Hotelling Rule⁶ support the assumption about strong influence of different externalities modifying the Hotelling Rule in the real economy and about relatively large absolute values of the modifier τ . Therefore, the government's policies with respect to extracting industries could be primary for the sustainable economic development. Concentration only on

⁵A book of D. Yergin (1991) is a good guide on the qualitative dependence of the world's economy on oil.

⁶The review is in (Gaudet, 2007).

the patterns of saving could be not enough. Dasgupta and Heal (1979, p. 309) wrote on this matter: “Governments of most countries ... have in the past been concerned with the rate of investment and, more recently, with the rate of utilization of the world’s exhaustible resources.” This implies the importance of opportunity to estimate the qualitative behavior of the long-run consumption along some “program” paths of extraction. The solution of this problem is in the following section.

4 The long-run consumption for any feasible scenario of extraction

Assume that the government is going to rely in the framework of the long-term energy program on some path of extraction which is recommended by the “oil peak” theorists or by some respectable institution.⁷ Then, the sustainability of growth in the economy will depend on 1) the possibility of realization of this path (reliability of the forecast) and 2) the consequences for the economy in the case that the path is realizable. I concentrate here only on the second question, namely, on the analysis of the long-run per capita consumption along some path $r(t)$ assuming that the path is realizable.

Per capita consumption in our framework is $c = q - \dot{k} = q(1 - w)$. Then, in order to examine the qualitative peculiarities of c , it is sufficient to analyze \dot{q} along the given scenario of extraction $r(t)$. The change of output in our model is $\dot{q} = f_k \dot{k} + f_r \dot{r} = \alpha q^2 w/k + \beta q \dot{r}/r$, which can be rewritten as $\dot{q} = (\alpha q^2 w/k) [1 + (\beta/(w\alpha))(k\dot{r}/(rq))]$. For simplicity, assume that $r(t)$ is monotone in the long run or, in other words, that condition $\int_0^\infty r(t)dt = s_0$ implies that $\dot{r} < 0$ for t big enough. This follows that $\dot{q} \geq 0$ in the long run

⁷See e.g. the scenario of the world oil extraction by the Cambridge Energy Research Associate (CERA, 2006).

iff

$$k |\dot{r}| / (rq) \leq w\alpha/\beta \quad (4)$$

This inequality contains the unknown path of capital $k(t)$ which can be defined from the differential equation $\dot{k} = wk^\alpha r^\beta$ (the saving rule). In general case (for any feasible $r(t)$) the solution of this equation cannot be expressed in elementary functions. However, the qualitative behavior of per capita consumption (or output) in the long run can be examined by considering the left hand side of inequality (4) in the limit with $t \rightarrow \infty$. Using L'Hôpital's rule we have

$$\begin{aligned} \lim_{t \rightarrow \infty} k |\dot{r}| / (rq) &= \lim_{t \rightarrow \infty} k^{1-\alpha} r^{-1-\beta} |\dot{r}| = \infty \cdot 0 = \lim_{t \rightarrow \infty} [k^{1-\alpha}] / \{1/(r^{-1-\beta} |\dot{r}|)\} \\ &= \infty/\infty = \lim_{t \rightarrow \infty} d[\cdot] / dt / d\{\cdot\} / dt \\ &= \lim_{t \rightarrow \infty} (1-\alpha)k^{-\alpha} \dot{k} / \{[(1+\beta)r^\beta \dot{r} |\dot{r}| - r^{1+\beta} d|\dot{r}|/dt] / \dot{r}^2\} \end{aligned}$$

which after substitution of the saving rule $\dot{k} = wq$ for \dot{k} becomes⁸

$$\begin{aligned} &w(1-\alpha) \lim_{t \rightarrow \infty} r^\beta \dot{r}^2 / [(1+\beta)r^\beta \dot{r} |\dot{r}| + \ddot{r}r^{1+\beta}] \\ &= w(1-\alpha) \lim_{t \rightarrow \infty} \dot{r}^2 / [\ddot{r}r - (1+\beta)\dot{r}^2]. \end{aligned}$$

This implies that the condition (4) can be reformulated as follows: $\dot{q} \geq 0$ iff $\lim_{t \rightarrow \infty} \dot{r}^2 / [\ddot{r}r - (1+\beta)\dot{r}^2] \leq \alpha / [\beta(1-\alpha)]$. After dividing the numerator and denominator of the left hand side by \dot{r}^2 , this condition becomes: $\lim_{t \rightarrow \infty} \ddot{r}r / \dot{r}^2 \geq 1 + \beta/\alpha$. The following Proposition 2⁹ summarizes the result.

⁸Note that unknown function $k(t)$ cancels out; and also that $d|\dot{r}|/dt < 0$ since for our case $\dot{r} \rightarrow -0$ with $t \rightarrow \infty$ and therefore $-d|\dot{r}|/dt = \ddot{r} > 0$.

⁹Proposition 2 generalizes the results obtained in (Bazhanov, 2007b; Andreeva and Bazhanov, 2007) for specific paths of extraction.

Proposition 2 *In the DHSS economy with the investment rule $\dot{k} = wq$ where $w \in (0, 1)$ and with the modified Hotelling Rule $\dot{f}_r/f_r = f_k + \tau(t)$ the growth of output q is sustainable in the long run ($\lim_{t \rightarrow \infty} \dot{q} \geq 0$) iff*

$$\Phi [r_\tau(t)] \geq 1 + \beta/\alpha$$

where $r_\tau(t)$ is smooth enough and $\Phi [r_\tau(t)] \equiv \lim_{t \rightarrow \infty} \ddot{r}_\tau r_\tau / \dot{r}_\tau^2$.

I will call Φ *the sustainability functional for the curve of extraction $r_\tau(t)$* and the value of $\Phi [r_\tau(t)]$ - *the sustainability number* of this curve.

Note that $\Phi [r_\tau(t)]$ does not depend on the saving coefficient w explicitly. However, this does not mean that the sign of \dot{q} does not depend on the pattern of saving at all in the long run. It would contradict with the known results. The equation (2) implies that any path $r(t)$ is the result of combined influence of the pattern of investment defined by w and the path of externalities $\tau(t)$ which includes the government's interventions.

It is easy to check this result using a classical example with $\tau \equiv 0$ (the standard Hotelling Rule) and with $w = \beta$ (the standard Hartwick Rule). Then the path of extraction (see e.g. Bazhanov, 2007b) is $r_{Hart}(t) = r_0 [1 + At]^{-\alpha/\beta}$ where $A = r_0\beta / [s_0(\alpha - \beta)]$. The first derivative is $\dot{r}_{Hart}(t) = -r_0 A \alpha [1 + At]^{-\alpha/\beta - 1} / \beta$ and the second is $\ddot{r}_{Hart}(t) = -r_0 A^2 \alpha (\alpha + \beta) [1 + At]^{-\alpha/\beta - 2} / \beta^2$ which follows $\ddot{r}_{Hart} r_{Hart} / \dot{r}_{Hart}^2 \equiv 1 + \beta/\alpha$. Proposition 2 implies for this case that per capita output and consumption are constant over time along the path r_{Hart} what coincides with the well-known result of J.M. Hartwick (1977).

The path r_{Hart} is monotonically decreasing starting with $\dot{r}_{Hart}(0) = -\alpha r_0^2 / [s_0(\alpha - \beta)]$ which is not observed yet in the real economy. Another evidence is that the prices for the different kinds of nonrenewable resources do not grow exponentially as it should be according to the standard Hotelling Rule (Gaudet,

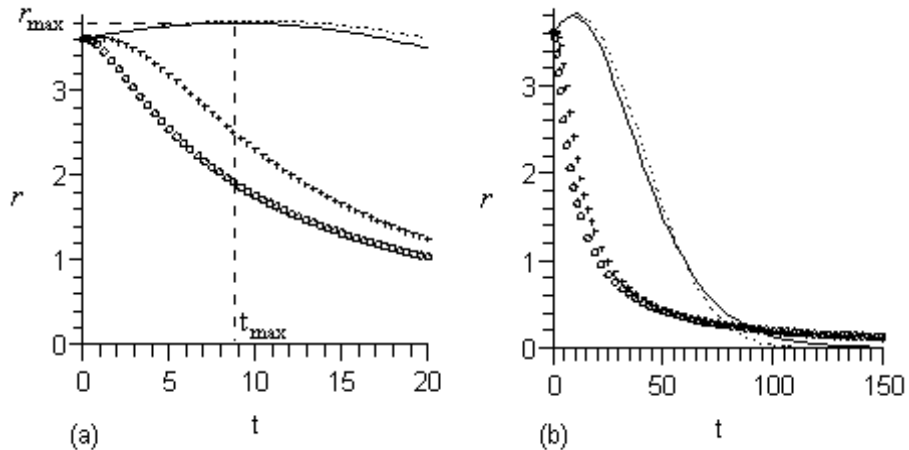


Figure 1: Paths of extraction [bln t per year; time t in years starting from 2008], (a) in the short run, (b) in the long run: the Hubbert curve (solid); the Gauss curve (dotted); the Cauchy curve (crosses); the rational curve (circles).

2007). This implies that the more realistic assumption is $\tau(t) \neq 0$. The following section provides the examples of calculating the sustainability numbers for some known paths of extraction that are compatible with the data from the real economy.

5 Consumption along the Hubbert and some other curves

There is a long-standing question about defining the “physical” peak of a non-renewable resource extraction. M.K. Hubbert (1956) and his followers (e.g. Laherrere, 2000) use a specific function with a single maximum (Hubbert curve) or a set of these functions, whose parameters are to be calibrated on the historical data of oil extraction and new fields discoveries. The curve(s) uniquely define the peak(s) and the rates of the future extraction. Laherrere

(2000) defines the Hubbert curve as follows:

$$r_H(t) = 2r_{\max} / \{1 + \cosh [b(t - t_{\max})]\}$$

where r_{\max} is the peak of extraction in a year t_{\max} . For the numerical example with the world oil extraction, $t_{\max} = 8.73$ and $r_{\max} = 3.7985$ (Fig. 1, solid line¹⁰). Parameter b defines the shape of the curve (deviation). Derivatives \dot{r}_H and \ddot{r}_H are

$$\dot{r}_H = -2br_{\max} \sinh [b(t - t_{\max})] / \{1 + \cosh [b(t - t_{\max})]\}^2$$

and

$$\ddot{r}_H = 2b^2r_{\max}(\cosh [b(t - t_{\max})] - 2) / \{1 + \cosh [b(t - t_{\max})]\}^2.$$

Then,

$$\ddot{r}_H r_H / \dot{r}_H^2 = \{\cosh [b(t - t_{\max})] - 2\} / \{\cosh [b(t - t_{\max})] - 1\}.$$

This implies that sustainability number for the Hubbert curve $\Phi [r_H(t)] \equiv \lim_{t \rightarrow \infty} \ddot{r}_H r_H / \dot{r}_H^2 = 1$ what is always less than $1 + \beta/\alpha$. This means that the path of consumption along this curve declines to zero in the long run (Fig. 2, solid line¹¹) regardless of the pattern of saving and the choice of parameters for the Hubbert curve.¹²

Hence, the government should do its best using taxes, regulations and education in order to shift the extracting industry from following this path.

¹⁰All the paths of extraction are calibrated on the current world's oil extraction data (World, 2007). The details of calibration are in Appendix 1.

¹¹The paths of consumption are obtained for all cases by solving numerically the differential equation for capital with $\alpha = 0.3$ and $\beta = 0.25$.

¹²This result coincides with the one obtained for the Hubbert curve in (Andreeva and Bazhanov, 2007).

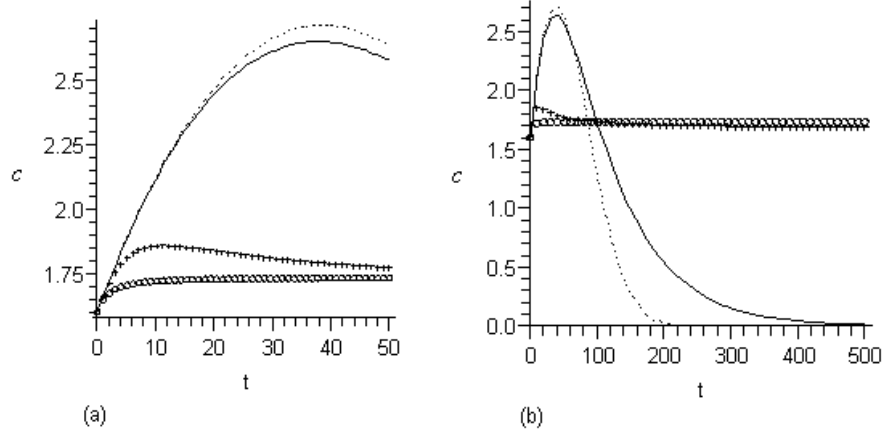


Figure 2: Paths of per capita consumption [time t in years starting from 2008], (a) in the short run, (b) in the long run along: the Hubbert curve (solid); the Gauss curve (dotted); the Cauchy curve (crosses); the rational curve (circles).

Another pattern of extraction considered in (Laherrere, 2000) is a well-known Gauss curve (Fig. 1, dotted):

$$r_G(t) = r_{\max} \exp \left[-(t_{\max} - t)^2 / 2b^2 \right],$$

where the roles of parameters are the same: t_{\max}, r_{\max} are the year and the amount of the maximum extraction and b describes the deviation. The derivatives are $\dot{r}_G = (t_{\max} - t)r_G/b^2$ and $\ddot{r}_G = [(t_{\max} - t)^2 - b^2]r_G/b^4$. Then the sustainability number is

$$\Phi [r_G(t)] \equiv \lim_{t \rightarrow \infty} \ddot{r}_G r_G / \dot{r}_G^2 = \lim_{t \rightarrow \infty} [(t_{\max} - t)^2 - b^2] / (t_{\max} - t)^2 = 1,$$

which implies the same pessimistic result as for the Hubbert curve (Fig.2, dotted).

The “optimistic” alternatives to the curves above can be found among the densities of the fat-tailed distributions. These paths of extraction can

be compatible with the Cobb-Douglas production function in a sense that they give the opportunity to sustain non-decreasing per capita consumption in the long run. This property is connected with the fat tail which provides more resources to the future generations. These patterns of the resource distribution make it possible to adjust capital by investing a fixed share of output adequately to the rate of shrinking of the essential resource. For example, the curve

$$r_C(t) = b^d r_{\max} / [b + (t_{\max} - t)^2]^d$$

is the probability density function for the Cauchy distribution for $d = 1$, where t_{\max} is the location parameter and b is the scale parameter.¹³ The generalizing parameter d is introduced here as a control variable for the sustainability number of this curve which is

$$\begin{aligned} \Phi [r_C(t)] &\equiv \lim_{t \rightarrow \infty} \ddot{r}_C r_C / \dot{r}_C^2 \\ &= 0.5 \lim_{t \rightarrow \infty} [(t_{\max} - t)^2 + 2d(t_{\max} - t)^2] / [d(t_{\max} - t)^2] \\ &= (1 + 2d)/2d. \end{aligned}$$

Proposition 2 implies that in the long run $\dot{q} \geq 0$ iff $(1 + 2d)/2d \geq 1 + \alpha/\beta$ or $d \leq \alpha/(2\beta)$.¹⁴ This curve with $d = \alpha/(2\beta)$ is depicted in Fig. 1 with crosses and the corresponding path of consumption (which is asymptotically constant as it must be for this value of d) is in Fig. 2 (crosses).

Another example of the extraction curve allowing for the sustainable economic development is the variant of the transition path with four parameters which I called “rational” and examined in (Bazhanov, 2007b). The

¹³ t_{\max} is not the expectation because the expectation and all other higher moments do not exist for this distribution due to the divergence of the corresponding integrals.

¹⁴This conclusion coincides with the result obtained in (Andreeva and Bazhanov, 2007).

first derivative of this curve is $\dot{r}_R(t) = (\dot{r}_0 + bt)/(1 + ct)^d$, the curve itself is $r_R(t) = r_0(1 + b_r t)/(1 + ct)^{d-1}$, where $b_r = c(d - 1) + \dot{r}_0/r_0$ and $\ddot{r}_R(t) = [b(1 + ct) - dc(\dot{r}_0 + bt)]/(1 + ct)^{d+1}$. The initial conditions imply $b = -c(d - 2)[r_0 c(d - 1) + \dot{r}_0]$ and then sustainability number is

$$\begin{aligned} \Phi[r_R(t)] &\equiv \lim_{t \rightarrow \infty} \ddot{r}_R r_R / \dot{r}_R^2 = r_0(1 + b_r t) [b(1 + ct) - dc(\dot{r}_0 + bt)] / (\dot{r}_0 + bt)^2 \\ &= 1 + 1/(d - 2) \end{aligned}$$

which means that the paths of consumption and production are not declining iff $d \leq \alpha/\beta + 2$. This coincides with the result of Corollary 1 (Bazhanov, 2007b). For comparison with the curve r_C , I considered r_R with $d = \alpha/\beta + 2$, which also implies asymptotically constant consumption (Figures 1 and 2, circles).

In the following section, I will use some of this examples in order to illustrate numerically the result of Proposition 1, namely, the roles of the Hotelling Rule modifier and the saving coefficient for the sustainability and the level of growth.

6 The paths of the Hotelling Rule modifier

Proposition 1 implies that the economy will follow decreasing (or increasing) path of consumption regardless of the value of the saving coefficient $w \in (0, 1)$ when the distortion $\tau(t)$ of the Hotelling Rule is not close enough to zero. This section shows how this result works in concrete cases using the extraction curves analyzed in previous section.

It was shown that the long-run consumption declines to zero along the Hubbert and Gauss curves for any patterns of saving. This means (Proposition 1) that in the long run the modifier $\tau(t)$ for these curves must be greater

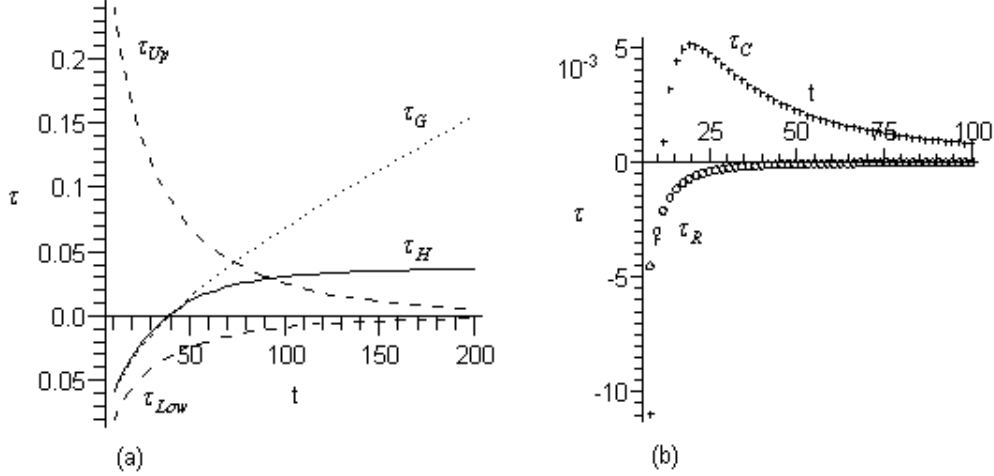


Figure 3: The paths of the Hotelling Rule modifiers for the paths of extraction: (a) Hubbert τ_H (solid); Gauss τ_G (dotted); (b) Cauchy τ_C (crosses); rational τ_R (circles).

than $f_k [1/\beta - 1] = \tau_{Up}$. One can see it in Fig. 3a where τ_H (τ for the Hubbert curve, solid line) asymptotically approaches a positive constant and τ_G (τ for the Gauss curve, dotted) goes to infinity while the upper (τ_{Up}) and the lower ($\tau_{Low} = -f_k$) bounds asymptotically converge to zero (dashed lines).¹⁵ The paths of consumption, declining to zero along the Hubbert curve, are depicted in Fig. 4a for the different values of w .

The paths of $\tau(t)$ for the Cauchy and the rational curves are rather deep inside the bounds τ_{Up} and τ_{Low} (Fig. 3b, the bounds are not depicted) and they converge with the bounds to zero regardless of the value of the saving coefficient. This implies the asymptotically constant consumption for all cases (Fig. 4b). The role of w is to define the level of the asymptote for the

¹⁵The bounds τ_{Up} and τ_{Low} are depicted only for the Hubbert curve in order not to overcrowd the figure. The behavior of these values for the Gauss curve is the same with the only difference that they approach zero faster.

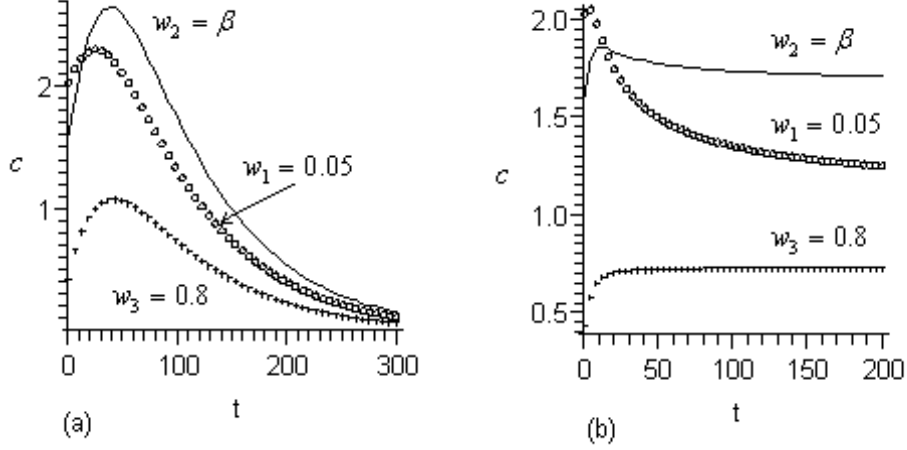


Figure 4: Paths of per capita consumption for the different saving coefficients along: (a) Hubbert curve; (b) Cauchy curve with $d = \alpha/(2\beta)$.

consumption path which one can see in Fig. 4b where the paths for $w_1 = 0.05$ and $w_3 = 0.8$ are the patterns of overconsumption and overinvestment correspondingly.

The paths of consumption that must grow in the long run according to Proposition 2 are depicted in Fig. 5 for the Cauchy curve with $d = \alpha/(2\beta) - 0.02$. The properties of this curve causing the long-run growth are illustrated in Fig. 6 in terms of observable variables (Corollary 1). Even for the pattern of overconsumption ($w_1 = 0.05$) there is a moment of time ($t_{\min} \approx 5000$ years, Fig. 6) when the ratio of the change of the price over the interest rate ($\left[\dot{f}_r/f_r \right] / f_k$) becomes equal to w_1/β which corresponds to the local minimum of per capita consumption and implies rather slow but unbounded growth for $t > t_{\min}$.

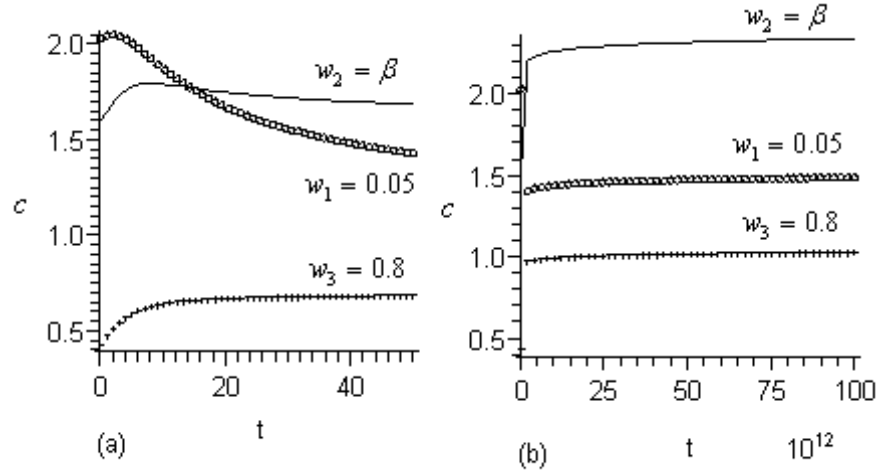


Figure 5: Paths of per capita consumption for the different saving coefficients along the Cauchy curve with $d = \alpha/(2\beta) - 0.02$ (long-run unbounded growth): (a) in the short run; (b) in the long run.

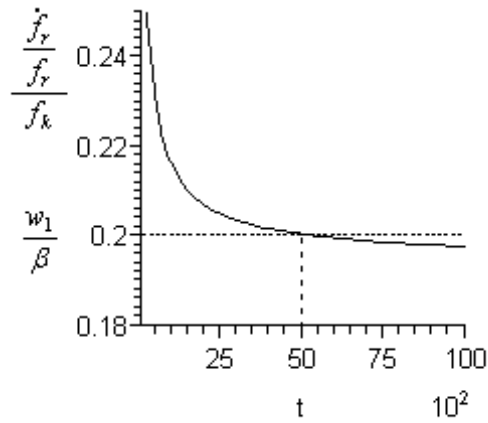


Figure 6: The path of the change-of-the-price-interest-rate ratio for the Cauchy path of extraction with unbounded growth of consumption in the long run (the saving coefficient $w_1 = 0.05$).

7 Concluding remarks

This paper have presented two new results for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model which was extended by the modified Hotelling Rule.

(1) Proposition 1 and Corollary 1 (Section 3) have shown that a resource-based economy is growing if and only if the Hotelling Rule modifier is no greater than the interest rate weighed by the factor $w/\beta - 1$ where w is the saving coefficient ($\dot{k} = wq$) and β is the resource elasticity; or, in terms of the observable variables (Corollary 1), the economy is growing if and only if the ratio of the change of the resource price over the interest rate is no greater than w/β . This result implies that the qualitative pattern of the economy's development (growth, stagnation, or decline) is defined by the path of the resource extraction that in turn is defined by the phenomena modifying the Hotelling Rule (including the government's policy). The pattern of investment (defined by w) specifies the level of consumption along the growing, constant, or declining path and defines the pattern of development in the cases when the Hotelling Rule modifier is close to zero.

(2) Proposition 2 (Section 4) offers a tool for estimating (weak) economic sustainability of some "program" path of a nonrenewable resource extraction. The easy-to-calculate "sustainability functional" offered in Proposition 2 allows for estimating the "sustainability number" for any smooth enough feasible path of the resource extraction. The sustainability number shows whether the pattern of consumption is growing, constant, or declining in the long run along this path of extraction. As an example, we¹⁶ have shown that the path of per capita consumption is always declining to zero in the long run along the well-known Hubbert curve regardless of the patterns of saving

¹⁶This result was obtained with Andreeva A.A in (Andreeva and Bazhanov, 2007).

and the choice of parameters for this curve. This is a warning sign appealing to the government's attention because the Hubbert curve is recognized in a large body of empirical research as a good estimation for the historical data of oil extraction. I offer an approach for constructing and some concrete alternative examples of the curves that allow for the oil-peak estimation and that are sustainable in a sense of nondecreasing consumption in the long run.

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8 Appendix 1

Calibration of the extraction curves

The parameters of the curves are calibrated on the current world's oil reserve and extraction data (World, 2007). The initial rate of extraction (on January 1, 2008) is $r(0) = r_0 = 3.61805$ bln t /year (1 t = 7.3 barrel), the paths are assumed to satisfy the feasibility condition $\int_0^\infty r(t)dt = s_0 = 182.4243941$ bln t (reserve estimate on January 1, 2008), and as $\dot{r}(0)$ I took the average $\dot{r}_0 = 0.04$ since 1984 (the methodology of estimation of \dot{r}_{0_i} for historical data is in Bazhanov (2006)). This way of calibration is essential for the feasibility of the paths of extraction while the conventional calibration on historical data usually lead to infeasible paths (Bazhanov and Vyscrebentsev, 2006).

a) *The Hubbert curve* (Andreeva and Bazhanov, 2007).

The initial value r_0 implies $r_{H \max} = 0.5r_0(1 + \cosh[-b_H t_{H \max}])$. The value of $\dot{r}(0)$ gives us $t_{H \max} = (1/b_H) \ln [(b_H r_0 + \dot{r}_0)/(b_H r_0 - \dot{r}_0)]$. Coefficient $b_H = (2r_0^2 + s_0 \dot{r}_0)/(s_0 r_0)$ is obtained from the feasibility condition $\int_0^\infty r(t)dt = s_0$.

b) *The Gauss curve*.

Using the condition $r(0) = r_0$ the curve can be expressed as follows $r_G(t) = r_0 \exp[t_G^2_{\max}/(2b_G^2)] \exp[-(t_G \max - t)^2/(2b_G^2)]$ which gives us $r_{G \max} = r_0 \exp[t_G^2_{\max}/(2b_G^2)]$. Initial condition for \dot{r}_0 implies $t_{G \max} = \dot{r}_0 b_G^2 / r_0$ and the feasibility condition for s_0 gives a nonlinear equation in b_G

$$(\sqrt{2\pi}/2)r_0 b_G \exp [\dot{r}_0^2 b_G^2 / (2r_0^2)] \left[1 + \operatorname{erf} \left\{ \dot{r}_0 b_G / (r_0 \sqrt{2}) \right\} \right] = s_0$$

with a single relevant root which I found numerically.

c) *The Cauchy curve*.

The peak of extraction $r_{C \max} = r_0 (b_C + t_{\max}^2)^d / b_C^d$ is expressed via r_0 , the initial condition for \dot{r}_0 is more convenient to use in this case for obtaining b_C

which gives us $b_C = 2r_0 t_{C_{\max}} d / \dot{r}_0 - t_{C_{\max}}^2$ and $t_{C_{\max}}$ is to be found from a nonlinear equation $\int_0^\infty r_C(t, t_{C_{\max}}) dt - s_0 = 0$.

d) Calibration of *the rational curve* is in (Bazhanov, 2007b).