

Errors-in-variables models: a generalized functions approach

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Abstract

Identification in errors-in-variables regression models was extended to wide models classes by S. Schennach (Econometrica, 2007) (S) via use of generalized functions. The problems addressed in (S) are revisited to extend and correct some of the results. Nonparametric identification here avoids decomposition of generalized functions into ordinary and singular parts, which may not hold, and gives a corrected proof. Continuity of the identification mapping and possible nonparametric estimation is discussed. Semiparametric identification via a finite set of moment conditions holds for classes of parametric functions; here such functions are explicitly characterized, the restriction in (S) that a moment generating function exist for the measurement error distribution is avoided; corrected weighting functions are provided.

Keywords: errors-in-variables model, generalized functions

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1 Introduction

The familiar errors in variables model with an unknown regression function, g , and measurement error in the scalar variable has the form

$$\begin{aligned} Y &= g(X^*) + \Delta Y; \\ X &= X^* + \Delta X, \end{aligned}$$

where variables X and Y are observable; X^* - mismeasured value and $\Delta X, \Delta Y$ are not observable. A widely used approach makes use of instrumental variables. Suppose that instruments are available and Z represents an identified projection of X on the instruments so that additionally $X^* = Z - U$; assume that U is independent of Z and

$$\begin{aligned} E[\Delta Y | Z, U] &= 0; \\ E[\Delta X | Z, U, \Delta Y] &= 0; \\ E(U) &= 0. \end{aligned}$$

These assumptions were made by e.g. Hausman et al. (1991) who examined polynomial regression. Newey (2001) added another moment condition for estimation in semiparametric regression leading to two equations for unknown g and F , the measurement error distribution, (all integrals over $(-\infty, \infty)$):

$$\begin{aligned} E(Y|Z = z) &= \int g(z - u)dF(u); \\ E(Y(Z - X)|Z = z) &= \int (z - u)g(z - u)dF(u). \end{aligned} \tag{1}$$

The functions g and F enter in convolutions; this motivates using Fourier transforms (Ft). Define $W_1(z) \equiv E(Y|Z = z)$; $W_2(z) \equiv E(Y(Z - X)|Z = z)$; Fourier transforms:

$$\begin{aligned}\varepsilon_y(\zeta) &= \int W_1(z)e^{i\zeta z} dz; & (2) \\ \varepsilon_{xy}(\zeta) &= \int W_2(z)e^{i\zeta z} dz; \\ \gamma(\zeta) &= \int g(x^*)e^{i\zeta x^*} dx^*; \\ \phi(\zeta) &= \int e^{i\zeta u} dF(u).\end{aligned}$$

Provided Fourier transforms are well defined, derivatives exist and the convolution theorem applies, (1) is equivalent to a system with two unknown functions, γ, ϕ :

$$\varepsilon_y(\zeta) = \gamma(\zeta)\phi(\zeta); \quad (3)$$

$$i\varepsilon_{xy}(\zeta) = \dot{\gamma}(\zeta)\phi(\zeta), \quad (4)$$

where $\dot{\gamma} = \frac{d\gamma}{d\zeta}$. S. Schennach (2007) (S) pointed out that these equations can be justified for a wide class of functions if one uses generalized functions; a sufficient condition is the following Assumption.

Assumption 1. $|g(x^*)|, |W_1(z)|, |W_2(z)|$ are defined and bounded by polynomials for $x^*, z \in R$.

Under Assumption 1 all the functions belong to the generalized functions space T' of *tempered distributions* (defined below). Fourier transform is a continuous invertible operator in T' , the convolution theorem applies there, all tempered distributions are differentiable in T' (thus $\dot{\gamma}(\zeta)$ is defined).

Denote by A the class of functions (g, F) , by A^* the class of functions (g) ; the mapping $P : A \rightarrow A^*$ is given by $P(g, F) = g$. Denote by B the class of functions (W_1, W_2) . Equations (1) map A into B ; Fourier transforms map B into $Ft(B)$, the class of functions that are Fourier transforms of functions

from B ; if equations (3,4) could be solved they would provide solutions $\phi^* = \phi I(\gamma \neq 0)$ (where $I(A) = 1$ if A is true, zero otherwise) and γ^* if $\phi \neq 0$; applying inverse Fourier transform would give $g^* = Ft^{-1}(\gamma^*)$. This sequence of mappings can be represented as follows:

$$\underset{(g,F)}{A} \rightarrow \underset{(W_1,W_2)}{B} \xrightarrow[\substack{Ft \\ (\varepsilon_y, \varepsilon_{xy})}]{Ft} Ft(B) \rightarrow \underset{(\phi^*, \gamma^*)}{Ft(\tilde{A})} \rightarrow \underset{(\gamma^*)}{Ft(A^*)} \xrightarrow[\substack{Ft^{-1} \\ (g^*)}]{Ft^{-1}} A^*. \quad (5)$$

If (5) provides the same result as P so that $g^* \equiv g$ (and $\gamma^* \equiv \gamma$) then g can be identified from the functions (W_1, W_2) with the identification mapping

$$M^* : B \rightarrow A^* \quad (6)$$

given by composition of the last five mappings in (5). The most challenging part is in solving the equations to establish the mapping (for $\gamma^* \equiv \gamma$)

$$S : \underset{(\varepsilon_y, \varepsilon_{xy})}{Ft(B)} \rightarrow \underset{(\phi^*, \gamma)}{Ft(\tilde{A})} \quad (7)$$

Two additional assumptions are made in (S).

Assumption 2. (i) Absolute moment of U exists and (ii) $\phi(\zeta) \neq 0$ everywhere on R .

Assumption 3. There exists a positive finite or infinite $\bar{\zeta}$ such that $\gamma(\zeta) \neq 0$ almost everywhere in $(-\bar{\zeta}, \bar{\zeta})$ and (ii) $\gamma(\zeta) = 0$ for all $|\zeta| > \bar{\zeta}$.

Under Assumptions 1-3 identification is possible as shown in Theorem 1 here; the theorem in (S) asserts an analytic formula (S, (13)) that relies on a decomposition that may not hold.

When the errors-in-variables problem is examined in the space of tempered distributions the corresponding (weak) topology is that of the space T' ; in that topology the mappings Ft , Ft^{-1} are known to be continuous, however, the mapping (7) may be discontinuous, rendering the identification mapping (6) discontinuous as well thus implying ill-posedness of the problem. The reason for this is that a too thin-tailed characteristic function may mask

high-frequency components in the Fourier transform of the regression function. Theorem 1 here provides a condition under which continuity obtains. When identification is provided by a continuous mapping consistent nonparametric estimation is possible as long as Fourier transforms of the conditional moment functions can be consistently estimated in T' ; this applies e.g. if the regression function is in the L_1 space; Corollary to Theorem 1 establishes this result.

Identification in classes of parametric functions requires that the mapping from the parameter space to the function space be (at least locally) invertible. (S) uses generalized functions to widen classes of parametric functions for which identification is provided by a finite number of moment conditions; in particular she expands classes of L_1 functions to which the results of Wang and Hsiao (1995) apply and also allows sums of such functions with polynomial functions, where before polynomial functions were considered only by themselves in Hausman et al. (1991). Her results rely on existence of a moment generating function for the measurement error and use special weighting functions (some of which are improperly defined). Here general classes of functions where such identification is achievable are explicitly characterized rather than via existence of moments conditions (S, Assumption 6), the requirement of a moment generating function for measurement error is avoided; appropriate weighting functions are given.

Section 2 deals with identification and well-posedness in the non-parametric case. Section 3 examines identification for the semiparametric model. Proofs are in the Appendix.

2 Non-parametric identification

In the first part of this section known results on generalized functions that confirm the existence and continuity of some of the mappings in (5) are provided, in particular, for the Fourier transform and its inverse. Other map-

pings, such as (7) require special treatment because they involve multiplication of generalized functions. Products in the space of tempered distributions cannot be generally defined (Schwartz's impossibility result, 1954, see also Kaminski and Rudnicki (1991) for examples) although there are cases when specific products are known to exist. Here the approach of Lighthill (1959) is extended to derive conditions under which some generalized functions can be multiplied by some continuous functions to obtain generalized functions. With this additional insight the existence and continuity of the mappings can be examined. In the second part of this section the identification result is proved and sufficient conditions for the identification mapping to be continuous are provided. A corollary that shows possibility of consistent (in metric of T') nonparametric estimation that in particular applies to functions in space L_1 completes this section.

2.1 Results about generalized functions and existence and continuity of mappings

All the known results in this section are in Gel'fand and Shilov's monograph (vol.1 and 2, 1964) - (GS) and in Lighthill (1959)- (L); they are listed for the convenience of the reader.

Definitions of generalized function spaces usually start with a topological linear space of well-behaved "test functions", G . Two most widely used spaces are $G = D$ and $G = T$. The linear topological space $D \subset C_\infty(R)$, where $C_\infty(R)$ is the space of all infinitely differentiable functions, consists of functions with finite support; convergence is defined for a sequence of functions that are zero outside a common bounded set and converge uniformly together with derivatives of all orders. The space $T \subset C_\infty(R)$ of test functions is defined as:

$$T = \left\{ s \in C_\infty(R) : \left| \frac{d^k s(t)}{dt^k} \right| = O(|t|^{-l}) \text{ as } t \rightarrow \infty, \text{ for integer } k \geq 0, l > 0 \right\},$$

$k = 0$ corresponds to the function itself; $|\cdot|$ is the absolute value; these functions go to zero faster than any power. A sequence in T converges if in every bounded region the product of $|t|^l$ (for any l) with any order derivative converges uniformly.

A generalized function, b , is defined by an equivalence class of weakly converging sequences of test functions in G :

$$b = \left\{ \{b_n\} : b_n \in G, \text{ such that for any } s \in G, \lim_{n \rightarrow \infty} \int b_n(t)s(t)dt = (b, s) < \infty \right\}.$$

An alternative equivalent definition is that b is a linear continuous functional on G with values defined by (b, s) . The linear topological space of generalized functions is denoted G' ; the topology is that of convergence of values of functionals for any converging sequence of test functions (weak topology); G' is complete in that topology. For $G = D$ or T the spaces are D' and T' , correspondingly. It is easily established that $D \subset T$; $T' \subset D'$ and that D' has a weaker topology than T' , meaning that any sequence that converges in T' converges in D' , but there are sequences that converge in D' , but not in T' . The space T' is also called the space of tempered distributions.

Any generalized function b in T' or D' is infinitely differentiable: the generalized function $b^{(k)}$ is the k -th order generalized derivative defined by $(b^{(k)}, s) = (-1)^k (b, s^{(k)})$. The differentiation operator is continuous in these spaces. For any probability distribution function on R^k the density function exists as a generalized function and continuously depends on the distribution function in the space G' , thus the generalized derivative of F , generalized density function f , is in T' .

Any locally summable (integrable on any bounded set) function $b(t)$ defines a generalized function b in D' by

$$(b, s) = \int b(t)s(t)dt; \tag{8}$$

any such function that additionally satisfies

$$\int (1 + t^2)^{-l} |b(t)| dt < \infty \text{ for some } l \geq 0. \quad (9)$$

similarly by (8) defines a generalized function b in T' . A distinction between functions in the ordinary sense (a pointwise mapping from the domain of definition into the reals or complex numbers) and generalized functions is that generalized functions are not defined pointwise. Generalized functions defined via (8) by ordinary function $b(t)$ are called regular functions; we can refer to them as ordinary regular functions in G' . The functions F, g, W_1, W_2 are ordinary regular functions in T' (and thus in D') by Assumption 1.

If a generalized function b is such that a representation (8) does not hold, it is said that b is singular, so any $b \in G'$ is either regular or singular. A well-known singular generalized function is the δ -function: $\delta : (\delta, s) = s(0)$. Any generalized function in D' or T' is a generalized finite order derivative of a continuous function. An ordinary function is regular if it integrates to a continuous function and singular otherwise. The function $b(t) = |t|^{-\frac{3}{2}}$, though ordinary in the usual sense, is singular in G' since it is not a derivative of a continuous function; it cannot be a regular ordinary function defining (8), in fact (see GS, v.1, p.51) it defines a generalized function by

$$(b, s) = \int_0^\infty t^{-\frac{3}{2}} \{s(t) + s(-t) - 2s(0)\} dt. \quad (10)$$

There is no known decomposition of the space of generalized functions into a space of generalized functions corresponding to ordinary functions and a space of singular functions; the pointwise argument provided in (S, Supplementary Material) is incorrect.

No special treatment is needed to consider complex-valued generalized functions; all the same properties hold. For $s \in T$ or D Fourier transform $Ft(s) = \int s(t)e^{it\zeta} dt$ exists and is in T . For $b \in T' : (Ft(b), s) = (b, Ft(s))$, so $Ft(b) \in T'$. Fourier transform defines a continuous and continuously in-

vertible linear operator in T' (but not for D'). Thus Fourier transforms of W_1, W_2, g , and of the generalized derivative, f , of F exist in T' and their inverse Fourier transforms coincide with the original functions. Since all the functions are differentiable as generalized functions $\dot{\gamma}$ exists in T' . By Assumption 2 the characteristic function ϕ is continuous, as is its derivative, ϕ' ; they are regular ordinary functions in T' .

Since G' does not have a multiplicative structure, products and convolutions can be defined for specific pairs only. When such products and convolutions exist the convolution theorem applies.

Convolution Theorem. If for $b_1, b_2 \in T'$, convolution $b_1 * b_2 \in T'$, product $Ft(b_1) \cdot Ft(b_2) \in T'$, then $Ft(b_1 * b_2) = Ft(b_1) \cdot Ft(b_2)$.

The definition of a product with a continuous function in T' , given in (L), specifies an infinitely differentiable function such that it and all its derivatives are bounded by a polynomial function at infinity; the proof of existence of the product requires verifying that the same limit obtains no matter which sequence is used for the generalized function. To consider products between individual functions where the continuous function may not be infinitely differentiable, the property that the product does not depend on the sequence that defines the generalized function has to be made a requirement. We thus say that ab for $b \in G'$ and continuous a is defined in G' if for any sequence b_n from the equivalence class of b there exists a sequence $(ab)_n$ in G such that for any $\psi \in G$

$$\lim \int a(x)b_n(x)\psi(x)dx \text{ exists and equals } \lim \int (ab)_n(x)\psi(x)dx. \quad (11)$$

Denote by 0_n a zero-convergent sequence that belongs to the equivalence class defining the function that is identically zero in G' .

Proposition *For the product ab between a continuous function a and $b \in G'$ to be defined in G' it is necessary and sufficient that (i) (11) hold for some sequence \tilde{b}_n in the class that defines b and (ii) for any zero-convergent*

sequence, $0_n(x)$,

$$\lim \int a(x)0_n(x)\psi(x)dx = 0. \quad (12)$$

Proof.

Any sequence b_n differs from a specific \tilde{b}_n by a zero-convergent sequence. ■

Lemma 1 *Under Assumptions 1-3 the products $\gamma\phi$ and $\gamma\dot{\phi}$ are defined in T' and in D' ; for $\tilde{\phi}^{-1} = \phi^{-1}(\zeta)I(|\zeta| < \bar{\zeta})$ the product $(\gamma\phi) \cdot \tilde{\phi}^{-1}$ is defined in D' ; the product $(\gamma\phi) \cdot \tilde{\phi}^{-1}$ is defined in T' if either $\bar{\zeta} < \infty$ or ϕ^{-1} is such that (9) holds for $b = \phi^{-1}$.*

Proof. See Appendix.

From Lemma 1 existence of products to justify the convolution theorem and thus equations (3,4) follows. Thus existence and continuity of all mappings in (5) with the exception so far of (7) follows. The mapping (7) involves solving equations (3,4) for the unknown functions and requires multiplication by ϕ^{-1} ; as one can see from Lemma 1 existence of such products in T' is not guaranteed; thus the proofs in (S) need to be amended.

2.2 The nonparametric identification theorem

This section contains two results. The first is Theorem 1 that proves the existence of the identification mapping M^* under Assumptions 1-3. It differs from the statement in (S, Theorem 1) in two ways: firstly, it does not rely on decomposition of generalized functions; secondly, it provides the condition for the continuity of the identification mapping and thus well-posedness of the problem. The second result is the corollary that shows that when continuity holds, consistent (in the metric of T') non-parametric estimation of the regression function is possible for some function classes, e.g. in space L_1 .

Theorem 1 *For functions satisfying Assumptions 1-3 the mapping M^* in (6) exists and provides identification for g ; if (9) holds for ϕ^{-1} or if $\bar{\zeta} < \infty$ the mapping is continuous; it can be discontinuous when (9) does not hold.*

Proof. See Appendix.

Remark. Theorem 1 actually holds under an Assumptions slightly more general than Assumption 1; it is sufficient to assume that the functions $|g(x^*)|, |W_1(z)|, |W_2(z)|$ satisfy (9).

The implication of this Theorem is that the identification result holds under the general assumptions 1-3. If ϕ is too thin-tailed, however, the mapping whereby the identification is achieved may not be continuous: this point is illustrated by the example in the proof of Theorem 1 where high frequency components b_n are magnified by multiplication with ϕ^{-1} from a thin-tailed distribution; this produces inverse Fourier Transforms that diverge.

The following Corollary provides sufficient conditions for consistent (in T') nonparametric estimation. Denote by $\rightarrow_{T'}$ convergence in topology of T' .

Corollary to Theorem 1 (a) *Under the conditions of Theorem 1 and (9) for ϕ^{-1} suppose that W_{1n}, W_{2n} are defined by (1) with some (unknown) functions g_n, F_n that satisfy Assumptions 1-3 and (9) for ϕ_n^{-1} ; for $\varepsilon_{1n}(\zeta) = Ft(W_{1n}), \varepsilon_{2n}(\zeta) = Ft(W_{2n})$ assume that $\varepsilon_{1n}(\zeta)$ is continuous and non-zero a.e.; $i\varepsilon_{2n}(\zeta) - \dot{\varepsilon}_{1n}(\zeta)$ is continuous and that*

$$\begin{aligned} \Pr(\varepsilon_{1n}(\zeta) \rightarrow_{T'} \varepsilon_y(\zeta)) &\rightarrow 1; \\ \Pr(\varepsilon_{2n}(\zeta) \rightarrow_{T'} \varepsilon_{xy}(\zeta)) &\rightarrow 1, \end{aligned} \tag{13}$$

then it is possible to find a sequence $g_n(x)$ such that $\Pr(g_n(x) \rightarrow_{T'} g(x)) \rightarrow 1$.

(b) *Suppose that the function $g(x) \in L_1$. Then there exist a sequence of step-functions, W_{1n} , and a sequence of piece-wise linear functions, W_{2n} , such that (13) holds and that (a) applies.*

Proof. See Appendix.

3 Parametric specification and identification

Semiparametric models with measurement error were examined for polynomial regression functions by Hausman et al (1991), for regression function in the L_1 space by Wang and Hsiao (1995). (S) significantly widened the classes of parametric models with errors-in-variables where identification can be achieved via moment conditions by utilizing generalized functions, but did not explicitly characterize the class of functions which she considered: verifying moment conditions of (S, Assumption 6) is needed. Here the class of parametric models is characterized directly in Assumptions 5 and 6; the assumptions give sufficient conditions for identification via moments. The results in (S) rely on existence of a moment generating function for the measurement error; this restriction is not imposed here. The results here correct the inappropriate weighting functions that rely on Definition 2 in (S); no non-zero function can satisfy that definition. Here as well as in (S) some moment conditions involve limits for sequences of weighting functions; the limits are explicitly given here.

Assumption 4. (i) The function $g(x^*)$ is in a parametric class of functions $g(x^*, \theta)$ where $\theta \in \Theta$; Θ is an open set in R^m ; for some $\theta^* \in \Theta$ equations (3,4) hold. (ii) There exists a (finite or infinite) $\bar{\zeta} > 0$ such that $\phi(\zeta) \neq 0$ for $\zeta < \bar{\zeta}$ and $\gamma(\zeta) \neq 0$ almost everywhere in $(-\bar{\zeta}, \bar{\zeta})$, where $\gamma(\zeta, \theta) = \gamma(\zeta, \theta^*)$ for all $\zeta \in (-\bar{\zeta}, \bar{\zeta})$ implies $\theta = \theta^*$.

If Assumptions 1, 2i, 4 hold, then by Theorem 1 γ is nonparametrically identified on $(-\bar{\zeta}, \bar{\zeta})$ and θ^* is identified. Recall that Fourier transform, $\gamma(\zeta)$, of a real function is a complex-valued function with the property that $\gamma(-\zeta) = \overline{\gamma(\zeta)}$ (where bar refers to complex conjugate). The following assumption requires existence of bounded $\bar{\zeta}$ in Assumption 4(ii) and in that region restricts the functions γ to have no more than a finite number of special points: Δ points of essential singularity and J of "jump" discontinuity. Notation $[x]$ is for integer part of x .

Assumption 5. The Fourier transform, $\gamma(\zeta, \theta)$, of the generalized function

$g(x^*, \theta)$ satisfies Assumption 4 (ii) with some $\bar{\zeta} < \infty$ and can be represented as

$$\gamma(\zeta, \theta) = \gamma_o(\zeta, \theta) + \gamma_s(\zeta, \theta), \quad (14)$$

where

(i) if $\Delta = 0$, $\gamma_s(\zeta, \theta) \equiv 0$; if $\Delta \geq 1$ and odd, there are singularities with support in points $s_0 = 0$, and $\pm s_l$, $l = 1, \dots, L (= \lceil \frac{\Delta}{2} \rceil)$, if $\Delta \geq 1$ and even, there are singularities with support in points $\pm s_l$, $l = 1, \dots, L (= \frac{\Delta}{2})$, thus

$$\gamma_s(\zeta, \theta) = 2\pi \sum_{l=0}^L \gamma_s(s_l, \theta), \quad \text{where} \quad (15)$$

$$\gamma_0(0, \theta) = \sum_{k=0}^{\bar{k}_0} \gamma_k(0, \theta) \delta^{(k)}(\zeta - s_l) \text{ for } S \text{ odd, zero otherwise;} \quad (16)$$

$$\gamma_s(s_l, \theta) = \sum_{k=0}^{\bar{k}_l} \frac{1}{2} (\gamma_k(s_l, \theta) \delta^{(k)}(\zeta - s_l) + \overline{\gamma_k(s_l, \theta)} \delta^{(k)}(\zeta + s_l)); \text{ for } l = 1, \dots, L;$$

(ii) $\gamma_o(\zeta, \theta) \in T'$ can be represented as an ordinary function that is continuous except possibly in a finite number of points and such that its generalized derivative, $\frac{d}{d\zeta} \gamma_o(\zeta, \theta)$, is of the form

$$\frac{d}{d\zeta} \gamma_o(\zeta, \theta) = \dot{\gamma}_{oo}(\zeta, \theta) + \dot{\gamma}_{os}(\zeta, \theta),$$

where if $J = 0$, then $\dot{\gamma}_{os}(\zeta, \theta) = 0$, and if $J > 0$, then for points b_j , $j = 1, \dots, \lceil \frac{J}{2} \rceil$, $\dot{\gamma}_{os}(b_j, \theta) =$

$$\gamma_{os0}(0, \theta) \delta(\zeta) I(J \text{ is odd}) + \sum_{j=1}^{\lceil \frac{J}{2} \rceil} \frac{1}{2} \left(\gamma_{osj}(b_j, \theta) \delta(\zeta - b_j) + \overline{\gamma_{osj}(b_j, \theta)} \delta(\zeta + b_j) \right),$$

$\dot{\gamma}_{oo}(\zeta, \theta)$ is an ordinary function continuous except possibly in a finite number of points;

(iii) $\gamma_o(\zeta, \theta) \neq 0$ except in a finite number of points in $(-\bar{\zeta}, \bar{\zeta})$;

(iv) At any non-zero singularity point: $s_l \neq 0$, $\gamma_o(\zeta, \theta)$ is continuous and non-zero.

Under Assumptions 1 and 5 g could be in L_1 , or a sum of a function from L_1 and a polynomial (singularity point $\zeta_0 = 0$) and also possibly a periodic function, e.g. $\sin(\cdot)$ or $\cos(\cdot)$ with singularities at some points $\pm s$, $s \neq 0$. In (S) singularities are considered at zero only. Here the parameters, $\gamma(\cdot, \theta)$, are allowed to take complex values, otherwise one would need to be more specific about the functions with singular Fourier transforms; since the functions are assumed known it is easy in each specific case to separate out the imaginary parts as in the case of polynomials.

Assumption 5 permits to write moment conditions; however, to ensure that the moment conditions provide identification of all parameters additionally the following Assumption 6 is made.

If $\Delta > 0$ define the matrices $\Gamma_y(s_l, \theta)$ and $\Gamma_{xy}(s_l, \theta)$ for each $s_l \geq 0$ (similarly to (S) for the case $s_l = 0$) by their elements:

$$\begin{aligned}\Gamma_{y,i+1,k+1}(s_l, \theta) &= \binom{k+i}{i} \gamma_{k+i}(s_l, \theta) I(k+i \leq \bar{k}_l); \\ \Gamma_{xy,i+1,k+1}(s_l, \theta) &= \binom{k+i+1}{i+1} \gamma_{k+i}(s_l, \theta) I(k+i \leq \bar{k}_l), \\ i, k &= 0, 1, \dots, \bar{k}_l.\end{aligned}$$

Denote by $\{A\}_{11}$ the first matrix element of a matrix A .

Assumption 6. The function γ satisfies Assumption 5. Additionally all $\gamma_o(\zeta, \theta)$, $\dot{\gamma}_{oo}(\zeta, \theta)$, $\gamma_s(s_l, \theta)$ are continuously differentiable with respect to the parameter, θ , in some neighborhood of θ^* . The $m \times 1$ parameter vector can be partitioned as $\theta^T = [\theta_I^T; \theta_{II}^T]$. For any component, θ_i , of $m_I \times 1$ vector θ_I (where $m \geq m_I \geq 0$) either

$$\gamma_o(\zeta, \theta^*) \frac{\partial}{\partial \theta_i} \dot{\gamma}_{oo}(-\zeta, \theta)|_{\theta^*} + \dot{\gamma}_{oo}(\zeta, \theta^*) \frac{\partial}{\partial \theta_i} \gamma_o(-\zeta, \theta)|_{\theta^*} \neq 0 \quad (17)$$

a.e., or if (17) does not hold for some i^* , then $\frac{\partial}{\partial \theta_{i^*}} \gamma_o(-\zeta, \theta)|_{\theta^*} \neq 0$. If $m_{II} > 0$ the matrix that stacks for all $s_l, l \geq 0$ matrices

$$\begin{pmatrix} \frac{\partial}{\partial \theta_{II}^T} [\Gamma_y(s_l, \theta)]^{-1} |_{\theta^*} \Gamma_y(s_l, \theta^*) + \frac{\partial}{\partial \theta_{II}^T} [\Gamma_{xy}(s_l, \theta)]^{-1} |_{\theta^*} \Gamma_{xy}(s_l, \theta^*) \\ \frac{\partial}{\partial \theta_{II}^T} \{ [\Gamma_y(s_l, \theta)]^{-1} |_{\theta^*} \Gamma_y(s_l, \theta^*) \}_{11} |_{\theta^*} \end{pmatrix}$$

is of rank m_{II} .

Assumption 7. The density function $p(z)$ exists and is positive.

Assumption 8. The characteristic function of measurement error, $\phi(\zeta)$, is \bar{k}_l times continuously differentiable at every s_l .

These Assumptions 7 and 8 are implicit in the proofs in (S).

We establish in Theorem 2 that moment conditions for the parameters θ of $\gamma(\zeta, \theta)$ hold and in Theorem 3 that the assumptions are sufficient for identification. The notation $\text{Re}(x)$ refers to the real part of a complex x .

Theorem 2. Under Assumptions 1,2(i), 4 (i), 7, 8

(i) if Assumption 5 (i,ii) holds there exist functions $r_y(z, \theta), r_{xy}(z, \theta)$ such that the moment

$$E \left(\frac{Y r_{xy}(z, \theta) + X Y r_y(z, \theta)}{p(z)} \right) \quad (18)$$

exists for θ in some neighborhood of θ^* and equals zero for $\theta = \theta^*$;

(ii) if 5(i-ii) holds there are functions $r_{y1n}(z, \theta)$ such that

$$\lim_{n \rightarrow \infty} E \left(\frac{Y r_{y1n}(z, \theta)}{p(z)} - 1 \right) \quad (19)$$

exists for θ in some neighborhood of θ^* and equals zero for $\theta = \theta^*$;

(iii) If $\Delta > 0$ and 5(i-ii) hold then for each $s_l \geq 0$ there exist vector functions $r_{ysl}(z, \theta), r_{ysl,n}(z, \theta), r_{xysl}(z, \theta), r_{xysl,n}(z, \theta)$, and a diagonal invert-

ible matrix M_l such that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \operatorname{Re} [\Gamma_y^{-1}(s_l, \theta) M_l^{-1} E \left(\frac{Y r_{ys,l,n}(z, \theta)}{p(z)} \right) \\ & + \Gamma_{xy}^{-1}(s_l, \theta) M_l^{-1} E \left(\frac{XY r_{xys,l,n}(z, \theta)}{p(z)} \right)] \end{aligned} \quad (20)$$

exists for θ in some neighborhood of θ^* and equals zero for $\theta = \theta^*$;

(iv) If $\Delta > 0$ and 5(i-iv) hold then for each $s_l \geq 0$ there exist functions $r_{ysl,1,n}(z, \theta)$, $r_{yslo,1,n}(z, \theta)$ such that for $s_0 = 0$

$$\lim_{n \rightarrow \infty} E \left(\frac{Y r_{ys0,1,n}(z, \theta)}{p(z)} - 1 \right) \quad (21)$$

and

$$\lim_{n \rightarrow \infty} \operatorname{Re} E \left(\frac{Y (r_{ysl,1,n}(z, \theta) - r_{yslo,1,n}(z, \theta))}{p(z)} \right) \quad (22)$$

exist for θ in some neighborhood of θ^* and equal zero for $\theta = \theta^*$.

Proof. See Appendix. Suitable functions $r.(z, \theta)$ are provided there.

Some of the moment conditions can be redundant. Different sets of weighting functions could be appropriate; similarly to reasoning in (S) the weighting functions are designed in a way that isolates different components of the γ function: the ones in (i) are for the ordinary function component and are supplemented by moments in (ii) for the case of a scale multiple for the ordinary component, the ones in (iii) are for the coefficients of the singular part with (iv) for the possible scalar factor at each singularity. If only (18) applies then the weighting functions proposed in (S) can be used, but for the other components the weights proposed here solve the problem without additional requirements that moment generating function for errors exist and avoid the problematic function in (S, Definition 2).

Define by $EQ(\theta)$ the vector with components provided by the stacked expressions (whichever are defined) from (18, 19, 20, 22).

Theorem 3. Under the conditions of Theorem 2 and Assumption 6 the

functions $r.(z, \theta)$ can be selected in such a way that the matrix $\frac{\partial}{\partial \theta} EQ(\theta^*)$ exists and has rank m .

Proof. See Appendix.

Theorem 3 provides sufficient conditions under which the equations $EQ(\theta) = 0$ fully identify the parameter vector θ^* .

By checking we can see that all the examples provided in (S) satisfy assumptions 5 and 6 here and thus sufficient conditions for identification hold. If the same parameters enter into both the ordinary and singular parts (S, assumption 6) may be violated, even though the moments can provide identification and the results here hold. An interesting example is the step-function $\gamma(\zeta, \theta) = \text{rect}(\zeta, \theta)$; the parameter of the corresponding regression function $g(x^*) = \text{sinc}(\frac{x^*}{\theta})$ (where $\text{sinc}(x) = \frac{\sin x}{x}$) cannot be identified from the moment conditions which are trivial: identically zero. The function violates assumption 6 here; in (S) one would need to check that $\frac{\partial}{\partial \theta} EQ(\theta) \equiv 0$ in violation of (S, Assumption 6).

4 Appendix

4.1 Proofs

Proof of Lemma 1.

By Assumption 1, γ and $\phi \in T' \subset D'$, by (3) $\gamma\phi \in T'$, and additionally (by applying the product rule to (3,4)) $\gamma\dot{\phi} \in T'$. Since $T' \subset D'$, the products are defined in D' as well. Now consider a sequence $(\gamma\phi)_n$ defined as follows: select some sequence $\tilde{\gamma}_n$ for γ from D ; then each γ_n has finite support; for a sequence of numbers $\varepsilon_n \rightarrow 0$ select $\tilde{\phi}_n$ in D such that $|\tilde{\phi}_n - \phi| < \frac{\varepsilon_n}{\sup|\gamma_n^{-1}\phi|}$ on compact support of γ_n . Then for $(\gamma\phi)_n = \tilde{\gamma}_n\tilde{\phi}_n$

$$\int \tilde{\gamma}_n\tilde{\phi}_n\phi^{-1}\psi = \int \tilde{\gamma}_n\psi + \int \tilde{\gamma}_n(\tilde{\phi}_n - \phi)\phi^{-1}\psi \rightarrow \int \gamma\psi.$$

Since $T \supset D$ this is valid in both spaces. The same applies to multiplication by $\dot{\phi}$. Now we check that (12) holds for $a = \phi^{-1}$. In D support of any ψ is bounded, on that compact set ϕ^{-1} is bounded thus (12) will hold and the product is defined in D' . If $\bar{\zeta} < \infty$ the product with $\phi^{-1}(\zeta)I(|\zeta| < \bar{\zeta})$ is similarly defined in T' . Now assume (9); consider for $\psi \in T$ the function $\tilde{\psi}(\zeta) = (1 + \zeta^2)^l \psi(\zeta) \in T$. Then

$$\begin{aligned} & \lim \left| \int \phi^{-1}(\zeta) 0_n(\zeta) \psi(\zeta) d\zeta \right| \\ &= \lim \left| \int \phi^{-1}(\zeta) (1 + \zeta^2)^{-l} 0_n(\zeta) \tilde{\psi}(\zeta) d\zeta \right| \\ &\leq \sup |(1 + \zeta^2)^{-l} \phi(\zeta)^{-1}| \lim \left| \int 0_n(\zeta) \tilde{\psi}(\zeta) d\zeta \right| = 0, \end{aligned}$$

since $\sup |(1 + \zeta^2)^{-l} \phi(\zeta)^{-1}| < \infty$ because $(1 + \zeta^2)^{-l} \phi(\zeta)^{-1}$ is continuous and integrable by (9). Thus the product with a zero-convergent sequence is zero and the product exists in T' ■

Proof of Theorem 1.

The proof makes use of different spaces of generalized functions and exploits relations between them. It proceeds in two parts.

First in part one, it is shown that from equations (3,4) the continuous function $\varkappa = \dot{\phi}\phi^{-1}$ can be uniquely determined on the interval $[-\bar{\zeta}, \bar{\zeta}]$ (where γ and consequently ε_1 differ from zero a.e. as generalized functions); this requires additionally considering the generalized functions spaces, D' and $D_0(U)'$ which is defined on the space of test functions that are continuous with support contained in U . The function $\tilde{\phi}$ is uniquely defined on the interval $[-\bar{\zeta}, \bar{\zeta}]$ as the solution of the corresponding differential equation that satisfies the condition $\phi(0) = 1$; define $\tilde{\phi} = \phi I(|\zeta| < \bar{\zeta})$; define $\tilde{\phi}^{-1}$ to equal $\phi^{-1} I(|\zeta| < \bar{\zeta})$. Of course, when $\bar{\zeta} = \infty$, $\tilde{\phi} = \phi$ and $\tilde{\phi}^{-1} = \phi^{-1}$ on R .

Next in part two, γ is defined as $\varepsilon_1 \tilde{\phi}^{-1}$. By Lemma 1 this product can always be uniquely defined as a generalized function in D' ; by construction $\gamma \in T' \subset D'$; this provides the required mapping M^* by applying inverse

Fourier Transform to γ . If either $\bar{\zeta} < \infty$ or ϕ satisfies (9) $\tilde{\phi}^{-1}$ satisfies (9) and the product $\varepsilon_1 \tilde{\phi}^{-1}$ is defined in T' ; in this case the mapping M^* is continuous. The proof concludes with an example that demonstrates that the mapping can be discontinuous if (9) does not hold.

Part one. Consider the space of generalized functions D' . By Assumption 2 ϕ is non-zero and continuously differentiable, then by differentiating (3), substituting (4) and making use of Lemma 1 to multiply by ϕ^{-1} in D' we get that the generalized function

$$\varepsilon_1 \phi^{-1} \dot{\phi} - (\dot{\varepsilon}_1 - i\varepsilon_2)$$

equals zero as a generalized function in D' . Denoting $\varkappa = \dot{\phi} \phi^{-1}$ we can characterize \varkappa as a continuous function in D' that satisfies the equation

$$\varepsilon_1 \varkappa - (\dot{\varepsilon}_1 - i\varepsilon_2) = 0. \quad (23)$$

If (23) holds in D' , it holds also for any test functions with support limited to $U : \psi \in D(U) \subset D$, and thus holds in any $D(U)'$.

We show that the function \varkappa is uniquely determined on $[-\bar{\zeta}, \bar{\zeta}]$ by (23) holding in $D(U)'$ for any interval $U \subset [-\bar{\zeta}, \bar{\zeta}]$. Proof is by contradiction. Suppose that there are two distinct continuous functions $\varkappa_1 \neq \varkappa_2$ that satisfy (23), then $\varkappa_1(\bar{x}) \neq \varkappa_2(\bar{x})$ for some $\bar{x} \in [-\bar{\zeta}, \bar{\zeta}]$; by continuity $\varkappa_1 \neq \varkappa_2$ everywhere for some interval $U \in [-\bar{\zeta}, \bar{\zeta}]$. Consider now $D(U)'$; we can write

$$\int \varepsilon_1 (\varkappa_1 - \varkappa_2) \psi = 0$$

for any $\psi \in D(U)$. A generalized function that is zero for all $\psi \in D(U)$ coincides with the ordinary zero function on U and is also zero for all $\psi \in D_0(U)$, where D_0 denotes the space of continuous test functions. For the space of test function $D_0(U)$ multiplication by continuous $(\varkappa_1 - \varkappa_2) \neq 0$ is

an isomorphism. Then from (23) we can write

$$0 = \int [\varepsilon_1(\varkappa_1 - \varkappa_2)] \psi = \int \varepsilon_1 [(\varkappa_1 - \varkappa_2)\psi]$$

implying that ε_1 is defined and is a zero generalized function in $D_0(U)'$. If that were so ε_1 would be a zero generalized function in $D(U)'$ since $D(U) \subset D_0(U)$; this contradicts Assumption 2. This concludes the first part of the proof since from \varkappa the function

$$\phi(\zeta) = \exp \int_0^\zeta \varkappa(\xi) d\xi$$

that solves on $[-\bar{\zeta}, \bar{\zeta}]$

$$\dot{\phi}\phi^{-1} = \varkappa; \phi(0) = 1$$

is uniquely determined on $[-\bar{\zeta}, \bar{\zeta}]$ and $\tilde{\phi}$ (and $\tilde{\phi}^{-1}$) defined above are uniquely determined.

Part two.

Consider two cases.

Case 1. Either $\bar{\zeta} < \infty$ or the condition (9) holds for $\tilde{\phi}^{-1}$. Multiplying $\varepsilon_1 (= \gamma\tilde{\phi})$ by $\tilde{\phi}^{-1}$ provides a tempered distribution by Lemma 1 here; it is equal to γ . The inverse Fourier Transform provides g . The theorem holds and moreover, since all the operations by which the solution was obtained were continuous in T' , the function g is recovered by a continuous mapping M^* .

Case 2. The condition (9) does not hold for ϕ^{-1} and $\bar{\zeta} = \infty$, so multiplication by $\tilde{\phi}^{-1} = \phi^{-1}$ may not lead to a tempered distribution. Consider now D' ; $T' \subset D'$. Multiplication by ϕ^{-1} is a continuous operation in D' ; define the same differential equations, solve to obtain ϕ and get via multiplication $(\gamma\phi) \cdot \phi^{-1}$ in D' the function $\gamma \in D'$. Since γ is the Fourier transform of g (a tempered distribution) it also belongs to T' , and it is possible to recover g by an inverse Fourier Transform.

In the following example the mapping M^* in (6) is not continuous. Define

β_n as a function with Fourier Transform equal to

$$b_n(x) = \begin{cases} e^{-n} & \text{if } n - \frac{1}{n} < x < n + \frac{1}{n}; \\ 0 \leq b_n(x) \leq e^{-n} & \text{if } n - \frac{2}{n} < x < n + \frac{2}{n}; \\ 0 & \text{otherwise.} \end{cases}$$

This $b_n(x)$ converges to $b(x) \equiv 0$ in T' . Indeed for any $\psi \in T$

$$\int b_n(x)\psi(x)dx = \int_{n-2/n}^{n+2/n} b_n(x)\psi(x)dx \rightarrow 0.$$

Suppose that $W_{1n} = W_1 + \beta_n$; from $b_n \rightarrow 0$ in T' and the continuity of the Fourier Transform mapping in T' , it follows that $\beta_n \rightarrow 0$ and $W_{1n} \rightarrow W_1$ in T' . Then $\varepsilon_{yn} = \varepsilon_y + b_n$. Suppose that ϕ is proportionate to e^{-x^2} . Then each $\gamma_n = \varepsilon_{yn}\phi^{-1}$ is in T' , the inverse Fourier transform, \tilde{g}_n , exists, but \tilde{g}_n does not converge to g in T' . Indeed, if it did so converge, then that would imply convergence $\gamma_n \rightarrow \gamma$ in T' , but $b_n(x)e^{x^2}$ does not converge in the space T' of tempered distributions. Define $\psi \in T$ by $\psi(x) = \exp(-|x|)$, then

$$\int_{n-2/n}^{n+2/n} b_n(x)e^{x^2}\psi(x)dx \geq e^{-n} \int_{n-1/n}^{n+1/n} e^{x^2-x}dx \geq \frac{2}{n}e^{-2n+(n-1/n)^2}.$$

This diverges. ■

Proof of Corollary.

(a) From arguments similar to those in the proof of Theorem 1 in the Appendix a continuous function $\varkappa_n(\zeta)$, that satisfies the equation

$$\varkappa_n(\zeta)\varepsilon_{1n}(\zeta) + (i\varepsilon_{2n}(\zeta) - \dot{\varepsilon}_{1n}(\zeta)) = 0 \quad (24)$$

in generalized functions, exists and is unique. Since all functions in (24) are continuous it holds for ordinary continuous functions and since ε_{1n} is

non-zero a.e. we have

$$\varkappa_n(\zeta) = (i\varepsilon_{2n}(\zeta) - \dot{\varepsilon}_{1n}(\zeta))(\varepsilon_{1n}(\zeta))^{-1}.$$

The generalized functions

$$\varkappa_n \varepsilon_{1n} - \varkappa \varepsilon_y = i(\varepsilon_{2n} - \varepsilon_{xy}) + (\dot{\varepsilon}_{in} - \dot{\varepsilon}_y) \text{ and } \varkappa(\varepsilon_y - \varepsilon_{1n})$$

converge to zero as generalized functions; as a result, so does $(\varkappa_n - \varkappa)\varepsilon_{1n}$, but since this is a continuous function this implies pointwise convergence and it follows from $\varepsilon_{1n} \neq 0$ that $\varkappa_n \rightarrow \varkappa$. From the differential equation $\phi_n^{-1} \dot{\phi}_n = \varkappa_n$ with the condition $\phi_n(0) = 1$ the function ϕ_n is uniquely determined; and $\phi_n \rightarrow \phi$ where $\phi^{-1} \dot{\phi} = \varkappa$, $\phi(0) = 1$. Then also since ϕ is non-zero, $\phi_n^{-1} \rightarrow \phi^{-1}$ pointwise; ϕ_n^{-1} satisfies (9) so that $\varepsilon_{1n} \phi_n^{-1}$ can be defined as a tempered distribution. Finally consider

$$\varepsilon_{1n} \phi_n^{-1} - \varepsilon_y \phi^{-1} = \varepsilon_{1n}(\phi_n^{-1} - \phi^{-1}) + (\varepsilon_{1n} - \varepsilon_y) \phi^{-1};$$

since the continuous function $\varepsilon_{1n}(\phi_n^{-1} - \phi^{-1}) \rightarrow 0$ and $(\varepsilon_{1n} - \varepsilon_y) \phi^{-1} \rightarrow 0$ in T' (as a tempered distribution) $\varepsilon_{1n} \phi_n^{-1}$ converges to γ in T' , and its inverse Fourier Transform converges to g as a tempered distribution (by continuity of inverse Fourier Transform in T').

(b) For any $g \in L_1$ there exists a sequence of step-functions $g_n \in L_1$ such that $\|g_n - g\|_{L_1} \rightarrow 0$ (implying $g_n \rightarrow_{T'} g$); for F there is a sequence of step-functions F_n such that $\sup |F_n - F| \rightarrow 0$ (implying $F_n \rightarrow_{T'} F$).

Specifically,

$$g_n(x) = \sum_{k=1}^N a_k I(b_k \leq x < b_{k+1}) \text{ for } b_1 < \dots < b_N;$$

$$F_n(x) = \sum_{j=1}^N c_j I(d_j \leq x) \text{ with } c_j > 0; \Sigma c_j = 1; d_1 < \dots < d_N,$$

implying $f_n(x) = \sum c_j \delta(x - d_j)$. Then $\phi_n(\zeta) = \sum c_j e^{i\zeta d_j}$. The function ϕ_n is not integrable (otherwise f would be continuous), thus ϕ_n^{-1} satisfies (9). All the parameters depend on n .

Then

$$W_{1n}(v) = \sum_{m=1}^{N^2} \alpha_m I(|v - t_m| < \delta_m);$$

$$W_{2n}(v) = \sum_{m=1}^{N^2} \alpha_m (v - \varepsilon_m) I(|v - t_m| < \delta_m),$$

where m corresponds to a pair (k, j) and $\alpha_m = a_k c_j$; $t_m = d_j + \frac{b_k + b_{k+1}}{2}$, $\varepsilon_m = d_j$, $\delta_m = \frac{b_{k+1} - b_k}{2}$. This represents W_{1n} as a step-function and W_{2n} as a piece-wise linear function. The conditional mean function W_1 can be consistently estimated in L_1 by step functions implying existence of a sequence $W_{1n}(v)$ such that $\Pr(W_{1n} \rightarrow_{T'} W_1) \rightarrow 1$, similarly, for some piece-wise linear $W_{2n}(v)$ $\Pr(W_{2n} \rightarrow_{T'} W_2) \rightarrow 1$ implying (13), moreover, we can write (using known Fourier transforms)

$$\varepsilon_{1n}(\zeta) = \sum_{k=1}^N 2\delta_k \alpha_k \chi_k(\zeta) \operatorname{sinc}\left(\frac{\delta_k \zeta}{\pi}\right);$$

$$\varepsilon_{2n}(\zeta) = -i \sum_{k=1}^N 2\delta_k \frac{d}{d\zeta} \left[\alpha_k \chi_k(\zeta) \operatorname{sinc}\left(\frac{\delta_k \zeta}{\pi}\right) \right] - \sum_{k=1}^N 2\delta_k \alpha_k \varepsilon_k \chi_k(\zeta) \operatorname{sinc}\left(\frac{\delta_k \zeta}{\pi}\right),$$

where the $\operatorname{sinc}(x)$ function is defined as $\frac{\sin \pi x}{\pi x}$ and $\chi_k(\zeta) = e^{it_k \zeta}$. The conditions about continuity and $\varepsilon_{1n}(\zeta)$ non-zero a.e., required in (a) are satisfied; (13) follows from the continuity of the Fourier transform operator in T' . ■

Prior to proof of Theorem 2 we make two preliminary observations.

Firstly, under Assumption 5 and 7 (that justifies products of γ_s and $\dot{\gamma}_s$ with ϕ) and by Lemma 1 equations (3,4) in T' lead to ($\mathbf{i}^2 = -1$):

$$\varepsilon_y(\zeta) = \varepsilon_{yo}(\zeta) + \alpha_s \varepsilon_{ys}(\zeta) \quad (25)$$

$$\text{with } \varepsilon_{yo} = \gamma_o(\zeta, \theta^*)\phi(\zeta); \varepsilon_{ys} = \gamma_s(\zeta, \theta^*)\phi(\zeta);$$

$$\mathbf{i}\varepsilon_{xy}(\zeta) = \mathbf{i}\varepsilon_{xyo}(\zeta) + \mathbf{i}\varepsilon_{xyos}(\zeta) + \alpha_s \mathbf{i}\varepsilon_{xys}(\zeta) \quad (26)$$

$$\text{with } \mathbf{i}\varepsilon_{xyo}(\zeta) = \dot{\gamma}_{oo}(\zeta, \theta^*)\phi(\zeta),$$

$$\mathbf{i}\varepsilon_{xyos}(\zeta) = \dot{\gamma}_{os}(\zeta, \theta^*)\phi(\zeta),$$

$$\text{and } \mathbf{i}\varepsilon_{xys}(\zeta) = \dot{\gamma}_s(\zeta, \theta^*)\phi(\zeta), \text{ where } \dot{\gamma}_s(\zeta, \theta) \text{ is the derivative of } \gamma_s(\zeta, \theta).$$

Second, to construct weighting functions some well-known functions are used. Denote by $T_R \subset T$ the space of test functions that are Ft of real-valued functions from T . A smooth cut-off (or "smudge") function is defined (e.g. in GS or L) as

$$f_{cut}(\zeta) = \exp\left(-\frac{1}{1-\zeta^2}\right)I(|\zeta| < 1);$$

"bump function" is

$$f_{bump}(\zeta) = \frac{f_{cut}(\zeta)}{\int_{-1}^1 f_{cut}(\zeta) d\zeta}.$$

For any $\varepsilon > 0$, interval $V = [a, b] \subset (a - \varepsilon, b + \varepsilon) = U$ the function (* stands for convolution)

$$f_{U,V}(\zeta) = I(a \leq \zeta \leq b) * f_{bump}\left(\frac{2\zeta}{\varepsilon}\right)$$

has the property that it equals 1 on V , 0 outside of U and takes values between 0 and 1.

For any $\xi \in R, p \geq 0, \varepsilon > 0$ consider a closed interval V_ξ that includes ξ , and some open $U_\varepsilon \supset V_\xi; U_\varepsilon = \{\zeta : |\zeta - \xi| < \varepsilon\}$ and the function $f_{U_\varepsilon, V_\xi}(\zeta)$, defined above. Define $f_{\xi,p}(\zeta) = (\zeta - \xi)^p f_{U_\varepsilon, V_\xi}(\zeta)$. This function has the property that

$$\frac{d^l f_{\xi,p,\varepsilon}(\zeta)}{d\zeta^l}(\xi) = \begin{cases} a_{\xi,p} \neq 0 & \text{if } l = p; \\ 0 & \text{otherwise.} \end{cases}$$

All the functions, f_{bump} , $f_{U,V}$, $f_{\xi,p,\varepsilon}$ are in T_R .

Proof of Theorem 2.

(i) Let ε be small enough that closed ε -neighborhoods of all the points of singularity and discontinuity of γ_o and $\dot{\gamma}_o$ do not intersect in $(-\bar{\zeta}, \bar{\zeta})$. Define by \bar{U}_ε^c the union of these ε -neighborhoods and $U = (-\bar{\zeta}, \bar{\zeta}) \setminus U^c$; then U is a finite union of open intervals; select in U a closed subinterval of positive measure and denote the union of those by V and consider $\mu(\zeta) \in T_R$ where $\mu(\zeta) = f_{U,V}(\zeta)$. Then $\frac{d^p \mu}{d\zeta^p}(s_l) = 0$ for all s_l, p ; ordinary functions $\dot{\gamma}_o(\zeta, \theta)\mu(\zeta)$ and $\gamma_o(\zeta, \theta)\mu(\zeta)$ are integrable for any θ . The inverse Ft's $r_y(z, \theta) = Ft^{-1}(\gamma_o(-\zeta, \theta)\mu(-\zeta))$ and $r_{xy}(z, \theta) = Ft^{-1}(\mathbf{i}\dot{\gamma}_o(-\zeta, \theta)\mu(-\zeta))$ exist. Since $\varepsilon_{yo}(\zeta) = \gamma_o(\zeta, \theta^*)\phi(\zeta)$, $\varepsilon_{xyo}(\zeta) = -\mathbf{i}\dot{\gamma}_o(\zeta, \theta^*)\phi(\zeta)$, $\dot{\gamma}_o(\zeta, \theta)$ and $\varepsilon_{xyo}(\zeta)$ are ordinary functions in T' and $\varepsilon_{yo}(\zeta)\mu(\zeta)$ and $\gamma_o(\zeta, \theta)\mu(\zeta)$ are continuous and satisfy (9), the products $\varepsilon_{yo}(\zeta)\dot{\gamma}_o(-\zeta, \theta)\mu(-\zeta)$ and $\varepsilon_{xyo}(\zeta)\gamma_o(-\zeta, \theta)\mu(-\zeta)$ are well defined in T' (in fact can be viewed as a product of a function in T' with a test function in T). Thus the integral

$$\int \left[\varepsilon_{yo}(\zeta)\mathbf{i}\dot{\gamma}_o(-\zeta, \theta)\mu(-\zeta) + \varepsilon_{xyo}(\zeta)\gamma_o(-\zeta, \theta)\mu(-\zeta) \right] d\zeta$$

exists. Since $\varepsilon_{yo}(\zeta) = \gamma_o(\zeta, \theta^*)\phi(\zeta)$, $\varepsilon_{xyo}(\zeta) = -\mathbf{i}\dot{\gamma}_o(\zeta, \theta^*)\phi(\zeta)$ the value of the integral is zero for $\theta = \theta^*$. Moreover, because the functions are zero together with all the derivatives at singularity points, ε_{yo} can be replaced by ε_y in the integral:

$$\int \left[\varepsilon_y(\zeta)\mathbf{i}\dot{\gamma}_o(-\zeta, \theta)\mu(-\zeta) + \varepsilon_{xy}(\zeta)\gamma_o(-\zeta, \theta)\mu(-\zeta) \right] d\zeta.$$

By Parceval identity this integral is

$$\int [W_1(z)r_{xy}(z, \theta) + W_2(z)r_y(z, \theta)] dz.$$

Multiplying and dividing by the non-zero function $p(z)$ does not change

the integral. Then by law of iterated expectations

$$\int \frac{1}{p(z)} E|_z (Y r_{xy}(z, \theta) + XY r_y(z, \theta)) p(z) dz = E \left(\frac{Y r_{xy}(z, \theta) + XY r_y(z, \theta)}{p(z)} \right).$$

This concludes the proof of (i).

(ii) By Assumption 5(iii) there exists a sequence $\xi_n \rightarrow 0$ such that $\gamma_o(\zeta, \theta) \neq 0$ for $\zeta : |\zeta - \xi_n| < \varepsilon_n < |\xi_n|$ and is continuous in those intervals; consider the function

$$\mu_n(\zeta) = \frac{1}{2} \left\{ f_{bump} \left(\frac{\zeta - \xi_n}{\varepsilon_n} \right) + f_{bump} \left(\frac{\zeta + \xi_n}{\varepsilon_n} \right) \right\}$$

The function $\frac{\mu_n(-\zeta)}{\gamma_o(-\zeta, \theta)}$ is a continuous function with bounded support. Set $r_{1yn}(z, \theta) = Ft^{-1} \left(\frac{\mu_n(-\zeta)}{\gamma_o(-\zeta, \theta)} \right)$. Then for any n we get

$$\begin{aligned} \int \varepsilon_{yo}(\zeta) \overline{\gamma_o(-\zeta, \theta)}^{-1} \mu_n(-\zeta) d\zeta &= \int \varepsilon_y(\zeta) \overline{\gamma_o(-\zeta, \theta)}^{-1} \mu_n(-\zeta) d\zeta \\ &= \int E(Y|z) Ft^{-1} \left(\overline{\gamma_o(-\zeta, \theta)}^{-1} \mu_n(-\zeta) \right) dz \\ &= E \left(\frac{Y r_{y1n}(z, \theta)}{p(z)} \right), \end{aligned}$$

where the first equality follows from the fact that $\varepsilon_{ys}(\zeta) \overline{\gamma_o(-\zeta, \theta)}^{-1} \mu_n(-\zeta) = 0$, the second by Parceval identity and the third by multiplying and dividing by $p(z) > 0$ and iterated expectation; the integral exists for each n . For θ^* we get $(\gamma_o(\zeta) = \overline{\gamma_o(-\zeta)})$

$$\begin{aligned} E \left(\frac{Y r_{y1n}(z, \theta^*)}{p(z)} \right) &= \int \varepsilon_{yo}(\zeta) \gamma_o(\zeta, \theta^*)^{-1} \mu_n(-\zeta) d\zeta = \int \phi(\zeta) \mu_n(-\zeta) d\zeta \\ &= \frac{1}{2} [\phi(\xi_n) + \phi(-\xi_n)] + O(\varepsilon_n) \end{aligned}$$

This converges to $\phi(0) = 1$.

(iii) Consider any $s_l \geq 0$. Below all relevant functions are subscripted by l . Define $\alpha_l : \alpha_0 = 1; \alpha_l = \frac{1}{2}$ if $l > 0$.

For ε as defined in (i) define the function $\mu_{l,i}(\zeta) = f_{s_l,i,\varepsilon}(\zeta) \in T_R$, then $\mu_{l,i}^{(i)}(0) \neq 0$, but $\mu_{l,i}^{(k)}(0) = 0$, $k = 0, \dots, i-1, i+1, \dots, \bar{k}+1$ and support of $\mu_{l,i}$ is in the interval $\zeta : |\zeta - s_l| < \varepsilon$; denote the derivative of $\mu_{l,i}$ by $\mu'_{l,i}$. For a sequence $\varepsilon_n \rightarrow 0$ consider $f_{U_n, V_n}(\zeta)$ for $U_n = \{\zeta : |\zeta - s_l| < \varepsilon_n\}; V_n = \{\zeta : |\zeta - s_l| < \frac{\varepsilon_n}{2}\}$ and define $\mu_{l,i,n}(\zeta) = \mu_{l,i}(\zeta) f_{U_n, V_n}(\zeta)$. The functions $\mu_{l,i}$ are in T_R . Denote by $r_{xys,l,i,n}(z)$ the inverse Ft: $Ft^{-1}(\mu_{l,i,n}(-\zeta))$ and by $r_{ys,l,i,n}(z)$ the inverse Ft: $Ft^{-1}(\mu'_{l,i,n}(-\zeta))$; they exist in T . The vector $r_{xys,l,n}(z)$ is defined to have $r_{xys,l,i,n}(z)$ as its i -th component; vector $r_{ys,l,n}(z)$ is defined similarly. Define by M_l the diagonal matrix with non-zero diagonal entries $\{M_l\}_{ii} = \mu_{l,i}^{(i)}(0) \equiv \mu_{l,i}^{(i)}(0)$, $i = 0, \dots, \bar{k}$.

Consider now the vector $(\varepsilon_y, \mu'_{l,n})$ with components $\int \varepsilon_y(\zeta) \mu'_{l,i,n}(-\zeta) d\zeta$ and $(\varepsilon_{xy}, \mu_{l,n})$ with $\int \varepsilon_{xy}(\zeta) \mu_{l,i,n}(-\zeta) d\zeta$, the integrals exist. Since the matrices $\Gamma_y(s_l, \theta)$, $\Gamma_{xy}(s_l, \theta)$ and M_l are invertible the expression

$$\Gamma_y(s_l, \theta)^{-1} M_l^{-1} (\varepsilon_y, \mu'_{l,n}) + \Gamma_{xy}(s_l, \theta)^{-1} M_l^{-1} (\varepsilon_{xy}, \mu_{l,n}) \quad (27)$$

is finite for every n . By Parceval identity

$$\int \varepsilon_y(\zeta) \mu'_{l,i,n}(-\zeta) d\zeta = \int W_1(z) r_{y,l,i,n}(z) dz,$$

thus by arguments similar to those in (i), (ii) this integral is $E \frac{Y r_{y,l,i,n}(z)}{p(z)}$ so that $(\varepsilon_y, \mu'_{l,n}) = E \frac{Y r_{y,l,n}(z)}{p(z)}$ and analogously $(\varepsilon_{xy}, \mu_{l,n}) = E \frac{XY r_{xy,l,n}(z)}{p(z)}$. We need to establish that limits as $n \rightarrow \infty$ exist. First, note that $\int \varepsilon_y \mu'_{l,i,n} - \int \varepsilon_{y0} \mu'_{l,i,n} = \int \varepsilon_{ys}(\zeta) \mu'_{l,i,n}(-\zeta) d\zeta$; then

$$\left| \int \varepsilon_{y0} \mu'_{l,i,n} \right| = \left| \int \varepsilon_{y0}(\zeta) f_{\zeta_l, i, \varepsilon}(-\zeta) f_{U_n, V_n}(-\zeta) d\zeta \right| \leq \max_U |\varepsilon_{y0}(\zeta) f_{\zeta_l, i, \varepsilon}(-\zeta)| 2\varepsilon_n$$

and goes to zero;

$$\int \varepsilon_{ys}(\zeta) \mu'_{l,i,n}(-\zeta) d\zeta = \alpha_l \sum_{k \geq i}^{\bar{k}} \gamma(s_l, \theta^*) (-1)^i \binom{k+i-1}{i-1} \mu_{l,i}^{(i)}(s_l) \phi^{(k-i+1)}(s_l) \quad (28)$$

and does not depend on n so the limit exists. By a similar representation for $-\int \varepsilon_{xys}(\zeta) \mu_{l,i,n}(-\zeta) d\zeta$ existence of (20) is established. For $\theta = \theta^*$ (27) leads to

$$\Gamma_y(s_l, \theta^*)^{-1} M_l^{-1}(\varepsilon_{ys}, \mu'_{l,n}) + \Gamma_{xy}(s_l, \theta^*)^{-1} M_l^{-1}(\varepsilon_{xys}, \mu_{l,n}) = 0.$$

Note that when $s_l \neq 0$ the same considerations apply to singularity at $-s_l$ with the difference that the $\Gamma(-s_l, \theta)$ matrices now are complex conjugate to $\Gamma(s_l, \theta)$. Combining provides the real part in (20).

(iv) Consider the first component of $\alpha_l^{-1} \Gamma_y(s_l, \theta)^{-1} M_l^{-1}(\varepsilon_{ys}, \mu'_{l,n})$ with $\mu'_{l,n}$, M_l defined in (iii); this component is $\alpha_l^{-1} \{\Gamma_y(s_l, \theta)^{-1} M_l^{-1}\}_{11} \int \varepsilon_{ys} \mu'_{l,1,n} d\zeta$. Note that $\mu'_{l,1,n} = \mu_{l,0,n}$, recall that $\mu \in T_R$. We see that $\phi(s_l)$ equals $\lim \alpha_l^{-1} \{\Gamma_y(s_l, \theta^*)^{-1} M_l^{-1}\}_{11} \int \varepsilon_{ys} \mu_{l,0,n} d\zeta$. Define

$$r_{ysl,1,n}(z, \theta) = \alpha_l^{-1} \{\Gamma_y(s_l, \theta)^{-1} M_l^{-1}\}_{11} F t^{-1}(\mu_{l,0,n}(-\zeta)).$$

Similarly to above by Parseval identity $\alpha_l^{-1} \{\Gamma_y(s_l, \theta)^{-1} M_l^{-1}\}_{11} \int \varepsilon_{ys} \mu_{l,0,n} d\zeta = E\left(\frac{Y r_{ysl,1,n}(z, \theta)}{p(z)}\right)$ and $\phi(s_l) = \lim E\left(\frac{Y r_{ysl,1,n}(z, \theta)}{p(z)}\right)$. For $s_0 = 0$ we have $\phi(0) = 1$. Thus (21) follows.

Consider now for $s_l \neq 0$ the function

$$\mu_{l,n}(\zeta) = \frac{1}{2} \left\{ f_{bump}\left(\frac{\zeta - s_l - \xi_n}{\varepsilon_n}\right) + f_{bump}\left(\frac{\zeta - s_l + \xi_n}{\varepsilon_n}\right) \right\}$$

similar to the one in (ii) and define $r_{slo,n}(z) = F t^{-1}\left(\frac{\mu_{l,n}(-\zeta)}{\gamma_o(-\zeta, \theta)}\right)$. For this function $E\left(\frac{Y r_{slo,n}(z, \theta)}{p(z)}\right) = \int \varepsilon_y(\zeta) \frac{\mu_{l,n}(-\zeta)}{\gamma_o(-\zeta, \theta)} d\zeta$ exists and at θ^* converges to $\phi(s_l)$. Thus (22) follows. ■

Proof of Theorem 3.

Let the vector $Q(z, \theta)$ denote the vector of functions for which expectations are taken in $E(Q)$; partition $Q(z, \theta)$ into $Q_I(z, \theta)$ corresponding to expressions in (18, 19) and $Q_{II}(z, \theta)$ for (20, 22). Then the matrix $\frac{\partial}{\partial \theta^T} E(Q(z, \theta))$ is a block matrix

$$\begin{pmatrix} \frac{\partial}{\partial \theta_I^T} E(Q_I(z, \theta)) & \cdot \\ \cdot & \frac{\partial}{\partial \theta_{II}^T} E(Q_{II}(z, \theta)) \end{pmatrix}$$

and it is sufficient to show that $\frac{\partial}{\partial \theta_I^T} E(Q_I(z, \theta))$ has rank m_I and $\frac{\partial}{\partial \theta_{II}^T} E(Q_{II}(z, \theta))$ has rank m_{II} .

For θ_I first note that interchange of differentiation with respect to the parameter and integration (taking expected value) for $\frac{\partial}{\partial \theta_I^T} E Q_I(z, \theta)$ follows from continuity in ζ of all functions in integrals of Fourier transforms and their continuous differentiability with respect to θ , so that $\frac{\partial}{\partial \theta_I^T} E(Q_I(z, \theta)) = E(\frac{\partial}{\partial \theta_I^T} Q_I(z, \theta))$. One can choose m_I functions μ defined in proofs of Theorem 2(i,ii) that are functionally independent and under Assumption 6 the corresponding m_I conditions of type $E(\frac{\partial}{\partial \theta_I^T} Q_I(z, \theta))$ will provide a rank m_I submatrix.

If for the functions μ in expressions $Q_{II}(z, \theta)$ in (iii,iv) of Proof of Theorem 2 the matrix $\frac{\partial}{\partial \theta_{II}^T} E(Q_s(z, \theta))$ has rank less than m_{II} consider varying the functions μ for all possible values of non-zero derivatives at the points s_l ; the rank cannot be deficient over all such choices without violation of Assumption 6. ■

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