Asset Location and Allocation with Multiple Risky Assets

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Abstract

Most working adults have access to a taxable brokerage account (TBA) and a tax deferred retirement account (TDRA). According to the existing literature, taxable bonds should be located in the TDRA, while equities should be located in the TBA due to the tax treatments of these accounts. If borrowing is not allowed mixed holdings can be optimal in either account but not in both simultaneously. But if borrowing is allowed all the wealth in the TDRA should be

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allocated to bonds. Unfortunately, the empirical findings are at odds with the theoretical predictions as investors do hold both equities and bonds in both of the accounts. This discrepancy is known as the asset location puzzle. In this paper, we revisit the asset location issue by extending the model to include multiple risky assets. This allows us to capture the interaction of portfolio diversification and the tax timing option, features that are not captured in the existing models. We find that the correlation structure of the risky assets and the borrowing constraints have substantial impact on asset location decisions. But this impact is sensitive to various levels of borrowing constraints. We also find that asset location and allocation decisions are sensitive to the Sharpe ratios of the risky assets, and the size of the retirement account. Consistent with the existing empirical findings, we also observe interesting mix of bonds and equities in both of the accounts with borrowing constraints and with reasonable relaxation of the borrowing constraints. Furthermore, we document the impact of borrowing constraints, and show that the assumption of unlimited borrowing is not innocuous. Failing to incorporate institutional restrictions on borrowing might have led to the theoretical inconsistencies.
1 Introduction

One of the most important decisions any working adult has to make pertains to the composition of his or her investment portfolio. In making investment decisions, she is faced with two major questions. The first question relates to asset allocation—how should she allocate her investment resources among various risky and risk free assets. The second question relates to asset location—given the availability of both taxable and tax-deferred savings accounts, how should she locate the risky and risk free assets in these two accounts.

There is a large literature addressing the asset allocation issue. But the issue of asset location has received less attention in the literature. In most of the portfolio optimization papers, taxes are nonexistent. However, taxes can have important effects on portfolio choice in real life. With differential tax treatments between the TBAs and the TDRAs, one asset (e.g., Microsoft stock) can now have multiple after-tax payoff structures. Consequently, the presence of tax-advantaged accounts influences the standard asset location problem. Investors not only must choose how much money to put in each asset, but also where those assets should be held.

Due to certain tax advantages in various tax deferred retirement accounts (TDRAs) as compared to taxable brokerage accounts (TBAs), taxes can play an important role in portfolio decisions. In a taxable account, investors have incentives to realize capital losses yet defer capital gains. This incentives imply that investors hold assets that have done well and sell those assets that have done poorly. Over time, following this strategy will lead to a poorly diversified portfolio. Thus, there is a trade off between the utilizing the tax timing option and maintaining a well-diversified portfolio.
In the last few years, several researchers have tried to address the portfolio location and allocation problem jointly with limited success. Results presented by the researchers are mixed, and models highly stylized and incomplete.\(^1\) Dammon, Spatt, and Zhang (2004), hereafter DSZ (2004), conclude that the optimal location for the taxable bonds is the TDRA, and the optimal location for equities is the TBA given that borrowing is not restricted. Using an arbitrage argument, Huang (2000) obtains similar results with deterministic liquidity shocks. The intuition driving their results is that assets with high yields (e.g., bonds) expose investors to larger tax burdens and should therefore be held in the tax deferred account. Moreover, by holding stocks in the TBA, the investor retains the tax timing option and can meet her liquidity needs without any penalty. In contrast to DSZ (2004) and Huang (2000), Shoven (1999) finds that hosting equity mutual funds in their tax-differed accounts may be optimal as opposed to hosting taxable corporate bonds in that location.\(^2\)

The findings of DSZ (2004) and Huang (2000) are at odds with much of the empirical evidence on asset location. Barber and Odean (2001) find mixed holding of stocks and bonds in the taxable accounts of the customers of a retail brokerage. The DSZ (2004) framework is unable to explain the mixed holdings, and the authors call this empirical finding as the *asset location puzzle* in DSZ (2004). These discrepancies between the empirical and theoretical findings imply either that the models being considered are incomplete, or an appreciable number of investors are making suboptimal investment decisions.

\(^1\)see Barber and Odean (2001)  
\(^2\)DSZ (2004) document some specific conditions under which Shoven (1999) conclusions are feasible within their framework.
In this paper, we revisit the asset location and allocation problem. The main difference in our analysis as compared to DSZ (2004) is that we consider multiple risky assets as opposed to a single risky asset. This seemingly simple extension can have important consequences on the optimal portfolio due to the incorporation of the potential of diversification. In particular, it is often the case that investors will choose to put a mix of equities and bonds in both the TBA and the TDRA for viable parameter values and borrowing restrictions. Moreover, one of the key contributions of our paper is the analysis of the relationship between location decision and the correlation structure of the risky assets.

Why does adding a second risky asset change the optimal portfolio so much? The answer is fairly simple. With a single risky asset, there is no concern about diversification. With multiple risky assets, the desire to defer gains and realize losses leaves an investor with a poorly diversified portfolio. By shifting some equities into the TDRA, the investor can freely trade these assets to maintain a diversified portfolio without incurring the tax consequences. The goal of the investor is to reduce the tax burden on the financial asset holdings as much as possible while at the same time maintaining a portfolio as diversified as possible. In part, these results are also due to more realistic treatment of constraints individual investors face on short sales and borrowing. Moreover, by considering multiple risky assets, we increase the value of the tax timing option, which is understated when a single market index is considered.

When investors are restricted from borrowing and short selling, they are unable to realize the full benefit of their tax timing options. We explore the impact that these restrictions can have on the optimal portfolio. From the casual observation of invest-
ment opportunities, institutional regulations, and industry practices, it is perceived that borrowing and short selling are not as readily available services for the average investors as have been presented in the literature. Moreover, whenever borrowing is allowed, the interest rates for borrowing and lending are not the same. Hence the liquidity needs cannot be met as easily with borrowing or selling of equities.

With multiple risky assets in the portfolio, any liquidation decision must take into account of the cost basis of the existing equities. If an investor needs a certain amount of cash, she may meet this need by liquidating any equity. But ignoring the cost basis of the equities may lead to extra tax burden. In terms of liquidation, high cost basis equities are preferred for liquidation over the low cost basis equities. Sometimes high return equities may be liquidated due to their high cost basis. So the cross-basis effects due to holding of multiple risky assets become very important with regards to liquidation for certain shocks. Liquidation may also be motivated by diversification concerns. For example, if a portfolio becomes heavily weighted in one risky asset, in order to maintain a desired level of diversification one may need to liquidate some of her position, and the liquidation decision is not independent of the consideration of the cost basis of other assets in the portfolio. Sometimes portfolio rebalancing for diversification purpose may reduce the value of the tax timing option. To get insight into all these issues we extend the existing models as discussed below and make qualitative conclusions.

To capture the diversification or portfolio rebalancing issues in a more pronounced manner, we consider two risky assets or equities, and a risk free taxable bond in our model. We capture the cross-basis effect by allowing for various parameterizations of
the basis of the risky assets. To capture the institutional restrictions such as borrowing and short selling, we consider three different cases, 1) constraints on both borrowing and short sales, 2) constraints on short sales but not on borrowing, and 3) constraints on none. We also consider various correlation structures in order to have specific insight into diversification. Limited analysis of bequest motive, retirement contribution limit, and optimal size of the retirement account is also conducted.

Our model predicts mixed holdings of equities and bonds in either the TBA or the TDRA under borrowing constraint when the equity prices are independent to each other. The level of the mix is dependent upon the retirement wealth ratio. But mixed holdings are not observed in both accounts simultaneously. When retirement wealth ratio is high, and borrowing is not allowed some equities may spill over to the TDRA due to the investor’s desire for higher (overall) risk exposure. These findings are consistent with DSZ (2004). The mix in the TDRA tends to disappear as the borrowing constraint is gradually relaxed. When the borrowing constraint is relaxed, we still observe mixed holdings of equities and bonds in the TDRA if the retirement wealth ratio is high. The level of borrowing required to have only bonds in the TDRA is exorbitant in some cases. Some of these results are sensitive to parameter values such as retirement wealth ratio. For example, to have all of the TDRA wealth in bonds, an investor with average retirement wealth ratio ($y = 0.4$) needs to borrow up to 100% of her TBA wealth whereas an investor with high retirement wealth ratio ($y = 0.7$) needs to borrow up to 175% of her TBA wealth. With borrowing constraints investors simply

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$3$Retirement wealth ratio is measured as the fraction of the total wealth that is held in the retirement account. More detailed description is given in the following sections, and the technical description is given in the Appendix.
can not achieve their desired risk exposure by holding only bonds in the TDRA. Due to lack of borrowing ability investors can not balance their long position in bonds in the TDRA with a comparable position in equities in the TBA. We observe relatively similar results when the short sale constraint is relaxed.

The correlation structure of the risky assets is a key determinant of the location decision. Under borrowing constraint, negative correlations generally lead to all equity allocation in the TDRA. But when borrowing constraint is relaxed up to 100%, mixed holding is observed in the TDRA for correlation coefficient of $-0.4$ and lower. Moreover, we notice that with multiple risky assets, the Sharpe ratios may be a key determinant of bond holding in the TBA under borrowing constraint. The bequest motive does not affect asset location decisions in substantial manner during the working age but does affect the investor’s consumption and equity holding during the retirement age. Retirement contribution limits do not affect location decisions but mildly affect allocation decisions in some cases. Furthermore, we observe a potential relationship between the optimal size of the retirement account and the borrowing constraint.

The rest of the paper proceeds as follows. In section 2, we discuss the existing models and their results, in section 3 we present the model and the methodology of our investigation, in section 4 we present the results, and we conclude in section 5.

2 Literature Review

As has been noted in Leland (2000), portfolio management entails two decisions. The investor should first specify the target or ideal portfolio strategy, which determines the
desired proportions of investment in different classes of assets such as stocks and bonds. The literature provides a rigorous framework for static and dynamic asset allocation in the absence of trading costs and other market frictions to facilitate the first decision (e.g., Markowitz (1952) and Merton (1971)). The second decision is how to implement the desired strategy. Issues to be considered in this implementation step include taxes, borrowing costs, market structure, financial regulations, trading costs, and costs related to other market frictions. Most of the papers in the portfolio optimization area consider the two-decision process separately, abstracting away from the dynamic interaction of the decisions and their outcomes. Our paper addresses both of the decisions and their interactions.

Constantinides (1983) introduced the idea of a tax timing option and pioneered the study of optimal investment and liquidation policy under capital gains taxes. According to the prescription in Constantinides (1983), investors should realize capital losses immediately and defer capital gains realization indefinitely. An important assumption is that there are no restrictions on short sales. In that case, the optimal portfolio choice is separable from the liquidation policy, since “overexposure” to an asset for tax timing reasons can be “undone” by offsetting short positions. A substantial amount of work has built around this basic foundation. Constantinides (1984) shows an application of the findings of Constantinides (1983). Dammon and Spatt (1996) derive the value of tax timing option with the key assumption that investors can circumvent the wash sale rules.\footnote{Current IRS rules restricts wash sales. The scope and enforcement of the rules has increased over time.} For further references on this strand of literature, interested readers may
consult Dammon and Spatt (1996).

Most of the existing literature on asset location and allocation focuses on an investor who has access to one risky asset and one risk free asset, a TBA, and a TDRA. The investor needs to decide in which assets she should invest, and where those assets should be hosted. The tax environment plays a major role in this decision process. The general result suggests that bonds should be held in the TDRA and equities should be held in the TBA. The following paragraphs explain the intuition of this analysis, using the arbitrage arguments presented by DSZ (2004).

Investors receive dividends from the investments in stocks and interest payments from their investments in bonds. Sale of stocks results in capital gains or losses. In the TBA, either type of income is taxed at the rate of the investor’s personal income tax rate ($\tau_d$) while capital gains are taxed at the capital gains tax rate ($\tau_g$). Suppose that the rate of interest ($r$) is greater than the dividend yield ($d_i$) of asset $i$, so for assets held in the taxable account, the tax burden on the income from bonds is greater than that on the dividend income from stocks ($r\tau_d > d_i\tau_d$). Taxes on capital gains (or losses) on equities in the TBA are due upon liquidation not upon accrual, so investors can avoid paying taxes by continuing to hold the equities. In contrast, the tax on income (interest and dividend payments) or on capital gains from equities can be deferred into the future (e.g., until retirement) if the assets are held in the TDRA. Moreover, due to the basis step-up provision, all the taxes may be forgiven at the death of the investor. As a result of these features of the accounts, the choice of asset location affects the after-tax payoff structure of the assets. These in turn may affect the asset allocation decision.
Assume that an investor is shifting one of her dollars from stock $i$ to bonds in her tax deferred account, which is offset by a shift of $x_i$ dollars from bonds to that stock in her taxable account. Denote the change in the net cash flow at the end of the year by $\Delta C_i$ then we have the following

$$\Delta C_i = r - [(1 + \tilde{g}_i)(1 + d_i) - 1] + x_i \{[(1 + \tilde{g}_i)(1 + d_i(1 - \tau_d)) - \tilde{g}_i\tau_g - 1] - r(1 - \tau_d)\} \tag{1}$$

where $\tilde{g}_i$ is the random capital gain. Taking the partial derivative of $\Delta C_i$ with respect to $\tilde{g}_i$ provides an expression, which does not change in value due to any change in $\tilde{g}_i$.

Setting this expression to zero and solving for $x_i$ gives

$$x_i = \frac{1 + d_i}{1 + d_i(1 - \tau_d) - \tau_g} \tag{2}$$

which can then be plugged back into (1), and after rearranging the terms we obtain

$$\Delta C_i = x_i \left[\frac{(r - d_i)(\tau_d - \tau_g)}{1 + d_i}\right] \tag{3}$$

Note that the expression of $\Delta C_i$ is free of $\tilde{g}_i$, and it represents a risk-free, after-tax cash flow that an investor can generate by simple relocation of the assets without incurring any cost. This is the basic intuition of the arbitrage argument.

Given that the term $\tau_d - \tau_g$ in (3) is positive,\footnote{In the case where $\tau_d = \tau_g$, the location of the assets does not matter.} it is clear that the sign of $\Delta C_i$ depends on the sign of the term $r - d_i$. Since the interest on a bond is generally higher than the dividend yield on a stock, this spread is typically positive. When $\Delta C_i$ is positive, the investor is strictly better off holding the bond in the tax deferred account and holding stock $i$ in the taxable account. Note that $\frac{\partial \Delta C_i}{\partial d_i} < 0$, hence $\Delta C_i$
is monotonically decreasing in $d_i$. Thus, for every asset where $r - d_i > 0$ the investor allocates bonds in the tax-deferred account, and stocks in the taxable account.

In general, the investor will always prefer to allocate her entire tax-deferred wealth to the asset with the highest yield, and allocate other assets in the taxable account. This basic principle can be used to decide on the location of any financial assets including various stock funds, mutual funds, taxable and non-taxable bonds. In DSZ (2004) further results are obtained by imposing restriction on borrowing. With this constraint, the investor first shifts her tax-deferred wealth into the asset with highest yield until no more of this shifting is possible. Then she allocates her remaining tax-deferred wealth to the asset with the next highest yield, and so on. The process continues with successively lower yielding assets until the investor's tax-deferred wealth is allocated completely. Then she makes offsetting adjustments in the taxable account for the desired level of risk exposure. But sometimes it may not be possible to make the offsetting adjustments due to the borrowing and short-sale restrictions. This may restrict her from having all bonds in the TDRA. Thus, with borrowing and short sale constraints, the investor may hold a mix of taxable bonds and equities in her tax deferred account, but only if the investor holds all equities in her taxable account.

Shoven and Sialm (2001) also address the asset location and allocation issue but they do so using considerably different modeling techniques. They use a continuous time approach but ignore the consumption/savings decision. They consider a broader menu of assets, including mutual funds and tax-exempt municipal bonds. According to them the preferred asset location is determined primarily by the tax rates facing the asset returns, and assets with the high tax rates should be allocated in the tax-
deferred account. Taxable bonds should be held in the tax-deferred account whereas
tax-exempt bonds should be held in taxable account. Stocks can be held in either
account depending on the tax-efficiency of the stock or stock portfolios.

The prescription of the existing theoretical models for investors to put taxable
bonds in the tax deferred account, and equities in the taxable account stands in sharp
contrast with observed investor behavior. DSZ (2004) considers this inconsistency
as the asset location puzzle. Using the data from the Survey of Consumer Finances,
Bergstresser and Poterba (2001) show that many households hold stocks in their tax-
deferred account but not in their taxable account. This suggests that substantial group
of households do not follow a “bond first in the tax-deferred account” asset location
strategy. In a similar study, Barber and Odean (2001) find mixed holdings of stocks and
bonds in both taxable and tax-deferred accounts (data sources are retail and discount
brokerages).

One possible explanation for the discrepancies between theoretical predictions and
empirical facts are the modeling assumptions. Under realistic restrictions on borrowing
and short selling, mixed holdings are feasible. As has been noted in DSZ (2004),
if investors are not allowed to borrow they may maintain mixed holdings. But the
prediction of all bond allocation in the TDRA with unlimited borrowing is not viable in
reality because of institutional restrictions on borrowing. For most individual investors
borrowing and short selling constraints are clearly binding. Diversification concerns
may also lead to mixed holding if the correlation structure of equities is amicable to
diversification. With good opportunities for diversification, mixed holdings may be
observed even without borrowing and short sale constraints. Superior value out of
diversification may outweigh the value of tax timing option. Moreover, the retirement wealth ratio of the investor in general may matter in portfolio choice. For example, consider an investor with high retirement wealth ratio and a preference for higher risk exposure who is facing borrowing and short selling constraints. Even though only bonds are optimally located in the TDRA, equities may spill over to the TDRA simply because her TBA account is too small, and she is unable to borrow to achieve her desired risk exposure through borrowing in the TBA.

Amromin (2001) offers an explanation for the empirical findings. His explanation is based on liquidity and the accessibility restrictions on assets in the TDRA. Investors would like to have access to their investment funds in the event of a negative income shock. Due to restrictions on accessing funds in the TDRA, investors have a preference for having some bonds in the TBA.

In this paper, we introduce a dynamic model with multiple risky assets to consider the diversification issue with regards to asset location and allocation decisions. We also consider the impact of various restrictions such as borrowing and short selling constraints on asset location and allocation. To the best of our knowledge, no one has considered the asset location and allocation issue with multiple equity assets yet. Most of the modeling has been done by using only one risky asset (generally interpreted as a well-diversified market equity portfolio) and a risk free bond. The absence of multiple risky assets restricts us from observing the interaction of the value of diversification and tax timing option. If the goal of the investor is to hold a well-diversified portfolio while maximizing the after-tax value of her investment within the tax and institutional constraints then the existing models are certainly limited in many ways. We capture
the impact of diversification by having multiple risky assets in our model. We also consider the robustness of our prediction to bequest motives. Our extension helps in explaining some of the inconsistencies between theoretical and empirical findings.

3 The Model

We consider a 20-year-old investor who works for 50 years, then lives in retirement for another 30 years before dying at age 100. The investor tries to smoothen her consumption, and wants to leave some of her wealth as a bequest. Here we assume that the investor and her descendant have the same preference over consumption and formulate the bequest as an $H$-period annuity, i.e. whatever amount the investor leaves for her descendant at her death is invested in an annuity, and the descendant receives the annuity payments over $H$ years. Thus the investor implicitly chooses the value of the bequest then she maximizes her utility over consumption.

The investor has access to two equities, and a taxable bond maturing in one year. Each equity price follows a binomial price process. The assets can be held in a TBA and/or a TDRA. The equity price processes are assumed to be independent of each other.\(^6\) During her working years, she receives labor or non-financial income of $L$ in each period. She contributes a fixed proportion of her labor income ($\alpha$) to her TDRA. But she is not allowed to withdraw any amount out of this account till retirement.\(^7\)

\(^6\)We consider the correlated cases in latter sections.

\(^7\)We make this simplifying assumption for the ease of our analysis. Moreover, given the penalty one has to pay in order to withdraw money from the tax-deferred account before retirement, it is unlikely that an average investor would want to do that in regular circumstances.
The investor seeks to maximize her utility of lifetime consumption by choosing seven control variables at each point in time. The control variables are the allocation toward consumption \((C_t)\), the number of shares of equity one and two held after rebalancing the TBA holdings \((n_{1t} \text{ and } n_{2t})\), amount of bonds \((B_t)\) held in the TBA, the fraction of retirement wealth held in equities \((\theta_{1t} \text{ and } \theta_{2t})\), and the fraction of retirement wealth held in bonds \((\theta_{3t})\) after rebalancing the TDRA. The rate of inflation is denoted by \(i\).

The formal optimization problem (with the constraints to be discussed later) is as follows:

\[
\max_{C_t, n_{1t}, n_{2t}, B_t, \theta_{1t}, \theta_{2t}, \theta_{3t}} \mathbb{E}_0 \left[ \sum_{t=0}^{T-1} \beta^t U \left( \frac{C_t}{(1+i)^t} \right) + \beta^T \sum_{j=1}^{H} \beta^j U \left( \frac{A_H W_T}{(1+i)^T} \right) \right]
\]

(4)

The objective function has two components. The first component represents the utility of the investor, defined over real consumption, throughout the working and retirement years (discounted at a rate of \(\beta\)). The second component represents the utility of the descendant out of the consumption of the bequest over \(H\) years (discounted at a rate of \(\beta\)). The ending period wealth is \(W_T\), and \(A_H\) is the \(H\)-period annuity factor. So \(A_H W_T\) is the annual consumption of the heir. The expectation is taken over the whole expression due to the random processes followed by the equity prices.

The maximization of the objective function, Equation (4), is subject to a number of constraints. We describe these conceptually here and refer the reader to the Appendix for a formal presentation.

\[\text{C1} \quad \text{Total wealth each period is the sum of the wealth in the the TBA (see C2) and...}\]

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8To maintain the clarity of the exposition, we describe most of the notation and definitions in the Appendix.

9In the Appendix, the constraints C1 - C9 is presented in order as Equations 6 - 14.
the TDRA (see C3) after subtracting the non-capital gains tax liabilities.

C2 Wealth in the TBA each period is the after tax-labor income plus the value of the holdings of the equities and bond after paying taxes on interest and dividends (before rebalancing the portfolio).

C3 Wealth in the TDRA each period is the prior-period ending balance times the TDRA portfolio gross return (no taxes are deducted).

C4 Consumption each period is the residual left after subtracting from total wealth (C1) the value of both investment accounts and any capital gains taxes in the TBA.

C5 The amount of money in the TDRA at the start of each period until retirement is the prior balance plus a fraction ($\alpha$) of the labor income.

C6 The amount of money in the TDRA at the start of each period in retirement is the prior balance less a withdrawal used for subsistence.

C7 Consumption must be non-negative in every period. Short selling is not allowed in the TBA.\textsuperscript{10} Borrowing is not allowed in the TBA.\textsuperscript{11} And investors hold non-negative amount of each asset in the TDRA.

C8 The investor must liquidate positions at death.

C9 Average cost basis is used.

\textsuperscript{10}We relax this constraint latter on to analyze the consequences of having this constraint.
\textsuperscript{11}We relax this constraint latter on to analyze the consequences of having this constraint.
We use the power form of the utility function with constant relative risk aversion coefficient $\gamma > 0$.

$$U(C_t) = \begin{cases} 
\ln(C_t) & \text{if } \gamma = 1 \\
\frac{C_t^{1-\gamma}}{1-\gamma} & \text{otherwise}
\end{cases}$$

This form of the utility function is chosen due to the convenience of being able to make conclusions that are independent of wealth level. This simplifies the analysis since we can ignore wealth as a state variable.

In the dynamic optimization problem, the investor observes a vector of state variables, $X_t = [P_{1t}, P_{2t}, P_{1t}^*, P_{2t}^*, n_{1t}, n_{2t}, W_t, Y_t, L_t]$. These variables contain information known at the time the investor chooses the control variables. As a modeling convenience, we follow DSZ (2004) and normalize the vector of state variables to obtain $x_t = [s_{1t}, s_{2t}, p_{1t}^*, p_{2t}^*, y_t]$\(^{12}\). The symbols $s_{kt}$, $p_{kt}^*$, and $y_t$ represent prior holding of stock $k$, $k = 1, 2$, as a fraction of the wealth in the TBA ($W_t$), basis-price ratio at time $t$, and wealth in the TDRA as a fraction of the beginning of the period total wealth ($W_t$) before trading at time $t$, respectively. The consumption wealth ratio is $c_t$.\(^{13}\) The variables $b_t$, $f_{1t}$, and $f_{2t}$ track the fraction of wealth in the TBA allocated to the risk free taxable bond, equity 1, and equity 2, respectively. The fraction of the TDRA wealth allocated to equity 1, equity 2, and the taxable bond are $\theta_{1t}$, $\theta_{2t}$, and $\theta_{3t}$, respectively. We assume that labor income or non-financial income is a constant fraction ($l$) of the total wealth during the working years and is zero in retirement.

\(^{12}\)This allows transformation of the optimization problem into a more tractable form. Please consult the Appendix for the normalization procedure and the reformulated optimization problem.

\(^{13}\) $c_t$ is the fraction of the total wealth allocated for consumption at time $t$. 

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3.1 Limitations and Potential Extensions

Certain limitations of the model need to be recognized. In our model, we have the labor or non-financial income as a constant fraction of the total wealth i.e. it does not follow any random process of its own. This assumption allows us to use the homogeneity of the objective function in reducing the dynamic optimization problem (we discuss this in the Appendix). Given the findings of Amromin (2001) and Huang (2000), we believe that introducing labor income shocks would strengthen our conclusions as it introduces other incentive to hold bond in the TBA. We use the average cost basis calculation rather than exact cost basis. In an exact cost basis calculation, the changing path of the cost basis is taken into account for each transaction for a given asset, and the cost basis is updated accordingly. The loss due to this simplification should be minor. DeMiguel and Uppal (2003) reports that the certainty equivalent loss from using the average tax basis instead of exact tax basis is less than 0.2% for problems with five periods and is less than 0.5% for problems with ten periods. Even though the loss is increasing in the time periods considered, the impact of it would be minor on our qualitative conclusions due to the magnitude of the optimized values of the relevant variables that we use to make our conclusions. Finally, we have a definite death date as opposed to a stochastic one. Overall, we believe that our main results would not change in any drastic manner due to any of the concerns we have mentioned here.
4 Results from Numerical Optimization

We solve the investor’s dynamic optimization problem numerically. The Appendix contains some technical details and a summary follows below.

To parameterize the model, we make the following assumptions.\textsuperscript{14} The agent is 20 at the beginning. Then she works for 50 years. Then lives in retirement for 30 years before dying at age 100. The equities have mean returns ($\mu$) of 9% and 13%, respectively. Corresponding standard deviations ($\sigma$) are 20% and 30%. Dividend rate for both of the stocks is 2%, and the interest rate is 6%. The tax rates are 36% for ordinary income and 20% for capital gains. The inflation rate is 3.5%.

The risk aversion parameter ($\gamma$) is set at 3. We set the fraction of total wealth earned as labor or non-financial income ($l$) at 15% at any given time during the working age. The portion of labor income saved in the retirement account ($\alpha$) is 20%. To capture the bequest motives we vary the bequest horizon parameter $H$. But for most of the analysis, we set $H$ at 20. In other words, in most of the cases we consider an investor who intends to leave a bequest amount that would allow her heir to have a fixed level of consumption for 20 years.

The numerical optimization is done over a grid of possible values for the state vector. We created a $5 \times 5 \times 5 \times 5 \times 8$ grid for the state space and used this grid for each of the 80 periods for the optimization ($s_{1t} \in [0.01, 1], s_{2t} \in [0.01, 1], p_{1t-1}^1 \in [0.01, 1.2], p_{2t-1}^2 \in [0.01, 1.2],$ and $y_t \in [0.1, .8]$). We have tried all parameterizations within the stated range but we report only a few interesting parametric outcomes. We solve the

\textsuperscript{14}We closely follow the parameter values from DSZ (2004) to make the comparison of the qualitative conclusions easier. DSZ (2004) documents viable reasons for selecting certain parameter values.
dynamic optimization problem by using backward induction and linear interpolation. We start with the terminal value of the value function of the investor. Then using the backward induction, we figure out in order to have that terminal value which asset location and allocation decisions the investor needs to make. We do this for each state vector in the grid. Once we cover the whole state space, we go backward one more period and repeat the same procedure. We continue this till we get to the initial period.

In the figures low value of parameters $s_{1t}$ and $s_{2t}$ (e.g., 0.1) indicate lower level of prior equity holdings, and high value of parameters $s_{1t}$ and $s_{2t}$ (e.g., 0.7) indicate higher level of prior equity holdings. If the basis-price ($p_{1t}^*$ or $p_{2t}^*$) ratio is above one, we have built in capital losses. When the basis-price ratio is below one, we have a built in capital gains, and when it is equal to one there are no capital gains or losses.

Here we focus specifically on parameter values that provide new insights related to the existing literature. Existing theoretical findings fail to explain the mixed holdings observed in the empirical data and ascribe this to suboptimal decision making in the part of the investors. Here we present our work in a way that may help us in understanding some of the reasons for these theoretical inconsistencies. The results in this section are organized as follows. First, we discuss the investor’s asset location and allocation decisions. Then we consider the issue of bequest motives. We discuss the retirement contribution limit, and its impact on location and allocation decision in the following subsection. In the following part of this section, we discuss the optimal size of the retirement account. At the end, we consider the investor’s consumption choices. For the ease of presentation, we start with the base portfolio and vary parameter values of the base portfolio in order to get other specific portfolios or cases. For
the base case we consider a portfolio that is heavily weighted in equities, both equities are equally weighted, and there are no built in capital gains or losses, \((s_{1t} = s_{2t} = .4, p_{1t-1}^* = p_{2t-1}^* = 1)\), please refer to Table 1 for details. To focus on the process of mixed holdings of stocks and bonds, we report combined equity allocation throughout this presentation.
4.1 Asset location and allocation

We observe interesting mix of bonds and equities in both of the accounts, the TBA and the TDRA, with and without borrowing constraint relaxation. The location and allocation decisions are sensitive to specific parameter values such as the basis-price ratios of the equities, the retirement wealth levels, Sharpe ratios of the equities, correlation structure of the equities, and the trading restrictions. We present the results in the following way. We first present asset location and allocation decisions for the base portfolio under various constraints. Then we present the asset location scenarios under various correlation structures. The relationship between retirement wealth ratio ($y$) and the TDRA equity holding under various levels of borrowing restriction is discussed at the end of this section.

Consider the base portfolio described earlier, see Table 1 for the set of parameter values. In this scenario, with borrowing and short sale constraints investors do not hold any bonds in the TBA irrespective of the retirement wealth ratio. We observe mixed holdings in the TDRA only for individuals with higher $y$ values. In panel (a) and (b) of Figure 1, we present the allocations of the total wealth of an investor with $y = 0.1$ in the TBA and the TDRA. Preferred location for equities is the TBA and for bonds it is the TDRA. Only around 10% of the total wealth is allocated to bonds but equity allocation is almost 80%, and equity allocation in the TBA increases during the retirement age. No bond is held in the TBA. Total allocation toward consumption and equities in the TBA is little less than 90% of the total wealth, see the line labeled Total. On the contrary, for investor with $y = .8$ allocations, presented in panel (c) and (d) of Figure 1, are different. There is a mix of bond and equity allocation in
the TDRA. Almost 65% of the total wealth is held in equities in the TDRA and this allocation is mildly declining in age. But less than 10% of the total wealth is allocated to the equities in the TBA during the working age. Equity holding is increasing in age in the TBA, the maximum is approximately 30%. Overall equity allocation seems to be negatively related to the retirement wealth ratio.

Now we consider wealth allocations across retirement wealth ratios ($y$) over time within the TBA and the TDRA. In this scenario, with borrowing and short sale constraints investors hold no bonds in the TBA irrespective of the retirement wealth ratio, see Figure 2 (c). Here equity allocation in the TBA is decreasing in retirement wealth ratio during the working years. The allocation to equity varies between 100 – 40%. That is investors invest 100 – 40% of their TBA wealth in Equities depending on their retirement wealth ratios. But equity holding is increasing in the retirement wealth ratio during the retirement age, see Figure 2 (a). For investors with high retirement wealth ratios, it is optimal to allocate most of their funds to equities in the TDRA for tax deferred growth whereas the allocation in the TBA is more suitable for immediate liquidity needs. Holding equities in the TBA allows them to meet their liquidity needs, capture the growth in capital gains, and leave the door open for exercising the tax timing option. As opposed to investors with higher retirement wealth ratios, for investors with lower retirement wealth ratios liquidity needs may be much stronger. So they tend to hold more stock in the TBA, and try to meet their liquidity needs as well as try to capitalize on the tax timing option. During retirement age bequest motive pushes the equity holding further up. This push may be due to the potential of capitalizing on tax the basis step-up or due to the availability of retirement funds that allows the
investor to spare more for investments. Notice in Figure 2 (a) for the investors with high retirement wealth ratios, the allocation towards equities during retirement years may go above 100%. The allocation is measured as a fraction of the wealth in the TBA at any given time. During the retirement age, the non-financial or labor income is zero but the investor receives returns from investments in the TBA, and a fraction of the TDRA wealth is also available from the retirement distributions. Given the bequest motive and reduction in gain from tax deferral during retirement, investors with high retirement wealth ratios may invest the extra money they receive from the TDRA distributions (amount available after consumption) in the TBA. Hence the relative investment amount in equity in the TBA may go beyond 100% even with the borrowing constraint for individuals with high $y$. The proportion of overall equity holding in the TBA tend to remain the same but the composition of the portfolio is affected by the tax basis and prior equity holdings. Given the opportunity of diversification, investors may opt for more equities given better risk adjusted return.

Sometimes location of bond in the TBA may be motivated by the inability to borrow to meet the liquidity needs. We notice an interesting relationship between the Sharpe ratio and bond holding. Up to certain threshold Sharpe ratio, bond allocation in the TBA is negatively related to the Sharpe ratios. We present this relation graphically in Figure 5 (a).

In the TDRA, bond holding is decreasing in retirement wealth ratio across time. For low retirement wealth ratio almost 100% of the retirement wealth is allocated to bonds. But it declines substantially over retirement wealth ratio, and it is below 50% for investors with high retirement wealth ratios, see Figure 2 (d). The equity
allocation is increasing in retirement wealth ratio in the TDRA, see Figure 2 (b). Here we obtain results similar to DSZ (2004). We have only equity allocation in the TBA and mix allocation in the TDRA. But mixed holdings in both accounts are not observed simultaneously.

When borrowing constraint is relaxed, we observe interesting changes in the asset location and allocation patterns. Investors with low retirement wealth ratio still prefer higher overall equity exposure as compared to the investors with high retirement wealth ratio. Borrowing is higher during the working age as opposed to the retirement age. But investors with high wealth ratio increase their equity holdings in the TBA, and decrease their equity holdings in the TDRA. For investors with low retirement wealth ratio, asset location decision does not change but the allocation to the equities increase in the TBA. When investors are allowed to borrow up to 100% of their TBA wealth, on average an investor with $y = 0.1$ borrows around 13.40\% of her total wealth during her working age, and borrows around 10.70\% of her total wealth during her retirement age. Equity holding increases in both part of the investors life, during working age it stands at 93.73\%, and during retirement age it stands at 97.56\%. But bond holding is relatively low, which stands at 10\% of the total wealth. And no bonds are held in the TDRA. These results are summarized in Table 2. The Magnitude column indicates the fraction of the wealth within a particular account allocated to a particular asset class or consumption. And the rest of the columns contain the frequencies of holdings of assets of particular asset class in different accounts over the investor’s lifetime. The two bottom rows present mean allocations for working age and retirement age.

Investors with higher retirement wealth make interesting location and allocation
decisions when they are allowed to borrow up to 100% of their TBA wealth. As opposed to the individuals with $y = 0.1$, investors with $y = 0.8$ borrow aggressively in the TBA, and increase their equity exposure. Interestingly, they reduce their exposure to equities in the TDRA, and increase it in the TBA. Whereas equity exposure decreases, bond exposure increases. On average equity allocation stands at 73.96% during the working age, and at 82.40% during the retirement age. Equity allocations in the TBA and the TDRA during the working age are 28.78% and 45.18%, respectively. In retirement age these allocations are 42.09% and 40.31%, respectively. Investors borrow aggressively, and borrow the maximum amount which is comparable to 20% of their total wealth. These results and holding frequencies are tabulated in Table 3.

Now we consider wealth allocations across retirement wealth ratios ($y$) over time within the TBA and the TDRA. Investors hold no bonds in the TBA irrespective of their retirement wealth ratios.\(^{15}\) Borrowing level is low for the investors with low retirement wealth ratios but investors with high retirement wealth ratios borrow heavily, see Figure 3 (c). The relaxed borrowing constraint is binding for investors with higher retirement wealth ratio, $y = 0.5 - 0.8$. Notice that for investors with lower retirement wealth ratio, $y = 0.1 - 0.5$, the constraint is not binding. Even though these investors could borrow more, they do not borrow to their maximum capacity.

When the borrowing constraint is relaxed, investors borrow heavily to capitalize on higher returns on the equities. The equity holding shifts upward, and the equity

\(^{15}\)Throughout the presentation, when we mention borrowing is not restricted we mean limited relaxation of the borrowing constraint. For the base case analysis, we allow the investor to borrow up to 100% of her wealth in the TBA. Unless otherwise stated, whenever we mention that there is no borrowing constraint, it should be understood that we are allowing the investor to borrow 100%.
holding may range from 100–250% of the wealth in the TBA depending on \( y \) and age. But the allocation pattern during working age and retirement age remains very similar to the constrained case, see Figure 3 (a). Given the relaxed borrowing constraint, investors with high \( y \) borrow heavily in the TBA to capitalize on higher returns from the equities. But the investors with lower level of retirement wealth are not as aggressive in borrowing (for them the borrowing constraint is not binding), they borrow money to hold more equity in the TBA but do not hold any equity in the TDRA. Their main allocation of funds in the TDRA is in bonds. By relaxing short sale constraints we obtain similar qualitative conclusions.

Now we consider various correlation structures of the equities using the base portfolio parameters. We consider equities with correlation coefficients \( (\rho) \) of \(-1, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, \) and 1. When the equities are perfectly positively correlated \( (\rho = 1) \) the equities do not serve much in terms of diversification. But close to 60% of the wealth in the TBA is held in equities before and after borrowing constraint relaxation to capitalize on the risk adjusted returns and the tax timing option. But as the magnitude of the correlation declines and the sign becomes negative the equity holding increases in the TBA. Interesting insights are obtained with regards to the TDRA. The location decision is influenced by the correlation structure of the equities. With negative correlation the value of diversification is high so investors always hold 100% in equities in the TDRA. But if \( \rho = 0 \) i.e. if the equity price processes are independent of each other, the equity holding is lower with borrowing constraints. When investors are allowed to borrow as much as 100% of their TBA wealth, they hold no equities in the TDRA if the correlation coefficient is positive (if \( \rho = 0.6 \) or \( \rho = 1 \)). But
if the correlation coefficient is negative all the retirement wealth is allocated towards equities. By relaxing the borrowing constraint further it is possible to have all in bonds in the TDRA but the level of borrowing is too high to be feasible. But when the equities are perfectly negatively correlated \( \rho = -1 \), it is not possible to have bonds in the TDRA even with high level of borrowing. The diversification benefit is simply too high. The diversification value outweighs the value of the tax timing option. In panel (a) and (b) of Figure 4, we present the allocations for equities with correlation coefficients \( \rho \) of \(-0.4, -0.2, 0, 0.2, 0.4 \) when borrowing is not allowed. In panel (c) and (d), similar allocations are presented when borrowing constraint is relaxed up to 100%. With borrowing constraint, it is always optimal to have 100% of the retirement wealth in equities when correlation coefficient are negative. But the equity allocation is lower when the equity returns are independent or positively correlated. When borrowing constraint is relaxed, equity location preference changes dramatically. For correlation coefficients of \(-0.2, 0, \) and 0.2 it is never optimal to host any equities in the TDRA, 100% of the TDRA wealth goes to bonds. Only for correlation coefficient of \(-0.4 \), it is optimal to hold around 20% of the TDRA wealth in equities.

One of the key determinants of bond location in the TDRA is the borrowing constraint. We document the interaction of retirement wealth ratio \( y \) and borrowing constraints in determining the bond holding in the TDRA for some specific parameter values. In panel (a) and (b) of Figure 6, we consider two scenarios. In panel (a), we have an investor with \( y = 0.4 \). This investor holds a mix of bonds and equities in the TDRA. But when she is allowed to borrow 100% of her TBA wealth, she holds only bonds in the TDRA. For making the point more clear, we also have an intermediate
case where we allow the investor to borrow only 25% of her TBA wealth. This shows the dependence of the investor’s location preferences on borrowing ability. The level of borrowing that is required to have all in bonds in the TDRA also depends on the retirement wealth ratio this can be observed in panel (b) where \( y = 0.7 \). Here the investor need to have much more borrowing ability in order to have total preference for bond in the TDRA. The investor needs to be allowed to borrow up to 175% of her TBA wealth to have all in bonds in her TDRA, which may deem unreasonable given the institutional restrictions. We have tried other scenarios where one equity got appreciated and other lost value; even in those scenarios we obtain similar results.

4.2 Bequest motive

The observations made above are mostly robust to changes in bequest motive, especially during the working years. The most noticeable difference that we observe is in the allocation for consumption, and equity holdings. We consider three specifications for this purpose with \( H = 5, 20, \) and 25. The investor is allowed to borrow up to 100% of her TBA wealth. The consumption level is around 10% of the total wealth during most of the working age but in the retirement age the consumption allocation changes. For low bequest motive, \( H = 5 \), we observe consumption increases dramatically over time whereas for higher bequest motive, \( H = 20 \) consumption is decreasing over time (see Figure 5 (b)). The growth in consumption for low bequest motive, \( H = 5 \), is pronounced in retirement age. Even with low bequest motive consumption allocation seems to be within the range of 10 – 15% of the total wealth. As the bequest motive increases consumption in the retirement age decreases sharply with age. When bequest
motive is low, the investor derives utility mostly out of her own consumption, hence she makes her consumption allocation decision accordingly. These conclusions seem to hold for variations in initial equity holdings. The magnitude of various allocations vary over bequest motives but the location decisions are qualitatively very similar in the working age. Variations in allocation strategy do arise in the retirement age. To provide some intuition, we consider the equity and bond allocation in the TBA for an investor with average retirement wealth, \( y = .4 \), holding the base portfolio. The allocations are presented graphically in panel (c) and (d) of Figure 5. In panel (c), the equity allocation in the TBA is considered. Equity holding in the retirement age declines for low bequest motive but increases for higher bequest motive. An investor with low bequest motive is focused in maximizing her own consumption within her life, so the value of the tax timing option or the benefit of the step-up clause is not as relevant to her. But for a person with higher bequest motive all these issues are important. Holding equity allows her to capitalize on the tax timing option and on the potential growth in capital gains. Panel (d) presents the bond allocation, and the bond allocation is uniform for all bequest motives. Heavy borrowing in the working age and reduction in borrowing in the retirement age. Investors tend to borrow heavily to capture the benefit of the higher returns of equity and on the deferral option. But in the retirement age investors start to dissave and borrowing is not as much needed to meet the liquidity need for consumption, or asset purchases. Equity holding is almost zero in the TDRA. All the wealth in the TDRA is allocated toward bond for growth with tax deferral.
4.3 Retirement contribution limit

Retirement contribution limit does not affect the asset location decision but within the accounts it affects the allocation decisions. To compare the changes in utility, we use certainty equivalent consumption. In the base case, retirement contribution level ($\alpha$) is set at 20%. In order to get insight into the issue, we keep all the parameter values for the base case the same except for $\alpha$. We vary the value of $\alpha$ to observe its impact on location and allocation decisions. With borrowing constraint, we observe minor reallocation of assets in the TBA, and all the changes in allocation take place during the working years. With higher value of $\alpha$, investors tend to reduce their equity holding in the TBA by $0 - 5\%$. Investors with the lower $y$ values benefit more from the relaxation of the contribution limit. But the gain in utility from the increase in the value of $\alpha$ is not strictly monotonically increasing, see Table 4. In column two of Table 4, we report the percentage changes in the certainty equivalent consumption due to the increase in the retirement contribution level from 5% to 20%. We observe that the investor with $y = 0.1$ benefits most and the investor with $y = 0.7$ benefits least from the increase in $\alpha$. We do a similar analysis without borrowing constraint in this case, the order of benefit received is maintained but the magnitude of the benefit increases, see column three of Table 4. Similar results are reported for the increase in $\alpha$ from 20% to 30% in column four and five of Table 4.

4.4 Optimal size of the retirement account

In our analysis, we have the retirement wealth ratio ($y$) as a state variable. This can be thought of as a proxy for the size of the retirement account. Any investor would want
to maximize her utility by allocating her entire wealth in the two accounts optimally. Here we present some analysis on this issue. We consider the base case parameters, and vary the $y$ values and try to observe the impact of these variations on utility, and measure these variations in terms of certainty equivalent consumption. In panel (c) of Figure 6, we present the case when borrowing is not allowed. The investors with higher $y$ values are better off overall which is quite intuitive. But the interesting case is presented in panel (d) where the investors are allowed to borrow up to 100% of their TBA wealth. We observe that optimality is not linearly dependent on the size of the retirement wealth ratio. Here the investor with $y = 0.6$ is better off than the investor with $y = 0.8$. So it is interesting to notice that there may be a key role of borrowing constraint in the determination of the optimal size of the retirement account.

4.5 Consumption

Consumption is relatively smooth across age and retirement wealth ratio. The level of consumption varies according to age, bequest motive, level of retirement wealth ratio, correlation structure of the risky assets, and restrictions on trading. Depending on various parameter values, the level of consumption ($c_t$) expressed as the fraction of the total wealth allocated toward consumption may vary between $3.5 – 13\%$. Our findings regarding consumption is mostly consistent with the existing findings in the qualitative sense.

We simulate for three different cases due to trading restrictions. In the first case, we don’t allow any borrowing and short selling, in the second case we allow borrowing, and in the third case we allow both borrowing and short selling. When borrowing
and short selling are allowed the investment opportunity set for the investors expands which allows the investors to consume at a higher level. We present the impact of the restrictions graphically for the investors with average retirement wealth ratio, \( y = 0.4 \), in Figure 7 (a). In this figure, \( C_1 \) represents the level of consumption with restrictions on both borrowing and short selling, \( C_2 \) represents the level of consumption when only borrowing is allowed, and curve \( C_3 \) represents the level of consumption under no restrictions. Level of consumption is lowest in \( C_1 \) but higher in \( C_2 \) and \( C_3 \). Consumption is decreasing in age irrespective of trading restrictions. Notice further that the gain in consumption from the relaxation of short sale constraint is not as substantive as it is in the case of borrowing constraint relaxation. The sharp decline in \( c_t \) during the working age is due to the fixed contribution that the investors make towards the TDRA out of their non-financial income. This fixed contribution increases the level of the total wealth of the investors over time. Since consumption level is relatively constant and the total wealth is growing over time, the value of \( c_t \) declines sharply during the working years. But the decline in \( c_t \) during the retirement age is relatively minor due to the absence of the fixed contribution of part of the non-financial income towards the TDRA. Rather than contributing to the TDRA, during the retirement age investors withdraw wealth from the TDRA for consumption purposes. The investors try to balance their consumption with their bequest motives. But \( c_t \) increases dramatically for the investors with low bequest motive. Since the investors are not leaving much for their heirs, they spend more of their total wealth for consumption. This intuition is graphically presented in Figure 5 (b).

In general, consumption is decreasing in age and mildly increasing in retirement
wealth ratio. Figure 7 (b) presents consumption across retirement wealth ratio over time under borrowing constraint. Here consumption is smooth across time and retirement wealth ratio. When borrowing constraint is relaxed the consumption surface shifts upward but the relative consumption patterns are maintained for all retirement wealth ratios (see Figure 7 (c)). Retirement wealth ratio affects consumption level but the impact is minor, in Figure 7 (d) we consider two retirement wealth levels to make this point. Here we consider investors who are not allowed to borrow. In Figure 7 (d), we present consumption profiles of two investors. The first investor has low retirement wealth ratio, $y = .1$, and the second investor has high retirement wealth ratio, $y = .8$. As can be observed from the figure, consumption level is higher for the investor with higher retirement wealth ratio.

Within any single period, consumption is relatively smooth across basis price ratio and prior equity holding. In any single period consumption level is dependent on the basis-price ratios of the equities and equity holdings. If equity holding level for either equities is high, the determinant of the consumption level is the basis-price ratio. If the basis-price ratio is low i.e., there is substantial built in capital gain then the consumption level would be low. But if there is built in capital loss consumption level may be higher. If there are built in capital losses, in the beginning of the period the investors will capitalize on the tax timing option i.e. they would claim the capital losses, and either consume more or rebalance their portfolios with the proceeds. Further, if prior equity holding is high, consumption level is higher in basis-price ratio. The consumption level of an investor with high equity holding with built in capital gains is relatively low compare to the investor with low equity holding with built in capital

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gains. The fraction of wealth that an investor with high level of prior equity holding with built in capital gains spends for consumption would certainly be smaller because of her high net wealth. But this outcome changes when basis-price ratio is greater than one.

5 Conclusion

Most of the modeling framework in the existing literature of asset location and allocation promote the view that by having larger coupon rates as compared to the dividend yield rates, an investor is much more exposed to tax burden by holding bonds in the taxable investment account (TBA) as opposed to holding it in the tax differed account (TDRA). Moreover, by holding stocks in the taxable account an investor can capitalize on the tax timing option, and meet her liquidity needs without any penalty. But the limitations of the existing models are substantial that begs careful reformulation and augmentation. In this paper, we introduce the issue of diversification in this discussion. And we find that correlation structure of the risky assets in the portfolio can be a key determinant of location decision. In many cases, we obtain results very similar to existing results. Specifically, we observe that DSZ (2004) conclusions under borrowing constraints do hold for viable parameter values with multiple risky assets. But we observe that when investors are allowed to borrow the existing conclusions would hold in some cases only under demanding assumptions on borrowing. Under borrowing constraints, negatively correlated assets would always be optimally allocated in the TDRA indicating the superiority of the value of diversification over the value of
tax timing option. Moreover, when the risky assets are perfectly negatively correlated hardly any assumption on borrowing would buttress the conclusions in DSZ (2004). We find interesting mix of equities and bonds in the TBA and in the TDRA with the borrowing constraint and with limited (viable) relaxation of the borrowing constraint, which is consistent with the empirical findings. Our analysis of bequest motives show that location decisions are robust to bequest motives. So in general, we can conclude that an average investor should take into account of the correlation structure of the risky assets in the portfolio along with borrowing constraints, retirement wealth ratio, basis-price ratio, and Sharpe ratios of the risky assets in making asset location and allocation decisions. Retirement contribution limit do not affect the location decision. We have touched upon the optimal size of the retirement account, but we believe a more thorough investigation would be fruitful in this area. Our limited analysis indicates that there may be an important relationship between the optimal size of the retirement account and the investor’s borrowing ability. Addressing some of the limitations and concerns that we have mentioned in the main body of the paper would certainly facilitate our understanding of asset location and allocation further. Most of the existing empirical data is aggregate level data. But more primary, and individual investor level data would certainly strengthen the empirical conclusions, and would allow us to test the quality of the theoretical models. For now our conclusions and analysis suffices as evidence in reconciling some of the concerns regarding the asset location puzzle.
6 Appendix

The investor solves the following optimization problem for asset location and allocation.\textsuperscript{16}

$$\max_{C_t,n_{1t},n_{2t},B_t,\theta_{3t}} E_t \left[ \sum_{t=0}^{T-1} \beta^t U \left( \frac{C_t}{(1+i)^t} \right) + \beta^T \sum_{j=1}^H \beta^j U \left( \frac{A_H \overline{W}_T}{(1+i)^T} \right) \right]$$

such that

$$\overline{W}_t = W_t + Y_t (1 - \tau_d), t = 0, \ldots, T$$

$$W_t = L_t (1 - \tau_d) + \sum_{k=1}^2 n_{kt-1} [1 + (1 - \tau_d) \alpha_k] P_{kt} + B_{t-1} [1 + (1 - \tau_d) \rho], t = 0, \ldots, T$$

$$Y_t = W_{t-1}^r \left[ \sum_{k=1}^2 \theta_{kt-1} (1 + \alpha_k) + \theta_{3t-1} (1 + \rho) \right], t = 0, \ldots, T$$

$$C_t = \overline{W}_t - \tau_g \sum_{k=1}^2 G_{kt} - \sum_{k=1}^2 n_{kt} P_{kt} - B_t - W_t^r (1 - \tau_d), t = 0, \ldots, T - 1$$

$$W_t^r = Y_t + \alpha L_t, t = 0, \ldots, Tr - 1$$

\textsuperscript{16}The objective function described in Equation 4 (in page-16) is the same as the objective function described in Equation 5. The constraints C1-C9 are presented as Equations (6)-(14), respectively.
\[ W^T_t = Y_t(1 - h_t), t = Tr, \ldots, T - 1 \] (11)

\[ C_t \geq 0, n_{kt} \geq 0, B_{kt} \geq 0, 0 \leq \theta_{it} \leq 1, t = 0, \ldots, T - 1, i = 1, 2, 3. \] (12)

\[ n_{kT} = 0, B_T = 0, W^T_T = 0 \] (13)

\[
P^*_kt = \begin{cases} 
\frac{n_{kt-1}P^*_kt-1 + \max(n_{kt} - n_{kt-1}, 0)P_{kt}}{n_{kt-1} + \max(n_{kt} - n_{kt-1}, 0)} & \text{if } P^*_kt-1 < P_{kt} \\
P_{kt} & \text{if } P_{kt-1} \geq P_{kt}
\end{cases}
\] (14)

\[ G_{kt} = \left\{ I(P^*_kt-1 > P_{kt})n_{kt-1} + [1 - I(P^*_kt-1 > P_{kt})]\max(n_{kt-1} - n_{kt}, 0) \right\} (P_{kt} - P^*_kt-1) \] (15)

Notations:

- \( C_t \): nominal consumption at time \( t \)
- \( B_t \): amount invested in bonds in the TBA at time \( t \)
- \( W_t \): wealth in taxable account after payment of the ordinary income taxes but prior to the payment of capital gains taxes at time \( t \)
- \( W^t \): total wealth at time \( t \)
- \( W^r_t \): wealth in the TDRA after contribution or withdrawal at time \( t \)
- \( Y_t \): pretax wealth in the tax-deferred account before contribution or withdrawal at time \( t \)
\[ L_t = \text{pretax non-financial or labor income in period } t \]
\[ \alpha L_t = \text{contribution to the retirement account from the pretax non-financial income at time } t \]
\[ h_t Y_t = \text{withdrawal from the retirement account at time } t \]
\[ P_{kt} = \text{price of stock } k \text{ at time } t \]
\[ n_{kt} = \text{number of the shares of stock } k \text{ held in the TBA} \]
\[ r = \text{nominal risk free interest rate} \]
\[ d = \text{nominal dividend yield} \]
\[ g_{kt} = \text{nominal pre-tax capital gain return from stock } k \text{ in period } t \]
\[ G_{kt} = \text{total realized capital gain from stock } k \text{ at time } t \]
\[ \tau_d = \text{income tax rate} \]
\[ \tau_g = \text{capital gains tax rate} \]
\[ H = \text{number of years for which the investor wants to leave funds for her descendant or bequest horizon} \]
\[ i = \text{inflation rate} \]

Inflation adjusted annuity factor:

\[ A_H = \frac{r^* (1 + r^*)^H}{(1 + r^*)^H - 1} \]

\[ r^* = [(1 - \tau_d) r - i]/(1 + i) \]

We reformulate the constrained optimization problem as a dynamic optimization problem. Then the value function of the dynamic optimization problem at time \( t \), \( V_t(X_t) \), is a function of the vector of the state variables, \( X_t \), at time \( t \).
\[ V_t(X_t) = \max \left\{ U\left( \frac{C_t}{1 + i^t} \right) + \beta E_t[V_{t+1}(X_{t+1})] \right\} \]

which is subject to the constraints listed in Equations 6 - 14.

\[ X_t = [P_{1t}, P_{1t-1}^*, P_{2t}, P_{2t-1}^*, n_{1t-1}, n_{2t-1}, W_t, Y_t, L_t] \]

We normalize the value function and introduce some new variables to simplify the problem further. Let
\[ v_t = \frac{V(X_t)}{[W/(1+i^t)]^{1-\gamma}}, \quad l = \frac{L_t}{W_t}, \quad s_{1t} = \frac{n_{1t-1}P_{1t}}{W_t}, \quad s_{2t} = \frac{n_{2t-1}P_{2t}}{W_t}, \]
\[ f_{1t} = \frac{n_{1t}P_{1t}}{W_t}, \quad f_{2t} = \frac{n_{2t}P_{2t}}{W_t}, \quad b_t = \frac{B_t}{W_t}, \quad c_t = \frac{C_t}{W_t}, \quad y_t = \frac{Y_t(1-\tau_d)}{W_t}, \quad w_t = \frac{W_t}{W}, \quad \delta_{kt} = G_{kt}/W_t, \quad \text{and} \quad p_{kt-1}^* = P_{kt-1}/P_t. \]

After the normalization, we deal with five state variables, beginning of the period equity 1 proportion or holding in the TBA \( (s_{1t}) \), beginning of the period equity 2 proportion or holding in the TBA \( (s_{2t}) \), basis-price ratio of equity 1 \( (p_{1t-1}^*) \), basis-price ratio of equity 2 \( (p_{2t-1}^*) \), and retirement wealth ratio \( (y_t) \) at time \( t \), respectively. We redefine the vector of state variables as \( x_t = [s_{1t}, s_{2t}, p_{1t-1}^*, p_{2t-1}^*, y_t] \). Note that \( l \) denotes the constant fraction of the total wealth that comes from the labor or non-financial income, \( L_t \). Retirement withdrawal rate is denoted by \( h_t \), and it is calculated by using the life expectancy table provided by IRS \( (h_t \text{ is the inverse of life expectancy at time } t) \).

For control variables, we have \( c_t, b_t, f_{1t}, f_{2t}, \theta_{1t}, \theta_{2t}, \text{ and } \theta_{3t} \), consumption-wealth ratio, fraction of the TBA wealth allocated to bonds after trading, fraction of the TBA wealth allocated to equity 1 after trading, fraction of the TBA wealth allocated to equity 2 after trading, fraction of the TDRA wealth allocated to equity 1, fraction of
the TDRA wealth allocated to equity 2, and fraction of the TDRA wealth allocated to bonds, respectively, at time $t$. Using the new notations, we can reformulate the constraints as follows.

\[
c_t = 1 - \tau_d \delta_t (1 - y_t) - (1 - y_t) \left( b_t + \sum_{k=1}^{2} f_{kt} \right) - w_t^r (1 - \tau_d) \tag{16}
\]

\[
R_{t+1} = \frac{b_t [1 + (1 - \tau_d) r] + \sum_{k=1}^{2} f_{kt} [1 + (1 - \tau_d) d_k] (1 + g_{kt+1})}{b_t + \sum_{k=1}^{2} f_{kt}} \tag{17}
\]

\[
R_{t+1}^r = \theta_{3t} (1 + r) + \sum_{k=1}^{2} \theta_{kt} (1 + d_k) (1 + g_{kt+1}) \tag{18}
\]

Equations (17) and (18) can be considered as the gross returns in the TBA and the TDRA. With the new notation and reorganization, we have the following reduced dynamic optimization problem

\[
v_t(x_t) = \max \left\{ \frac{c_t^{1-\gamma}}{1 - \gamma} + \beta E_t [v_{t+1}(x_{t+1}) w_{t+1}^{1-\gamma}] \right\} \tag{19}
\]

such that

\[
w_{t+1} = \left( \frac{R_{t+1}}{1 + i} \right) \left( b_t + \sum_{k=1}^{2} f_{kt} \right) \frac{1 - y_t}{1 - l(1 - \tau_d)} + \left( \frac{R_{t+1}^r}{1 + i} \right) \frac{w_t^r (1 - \tau_d)}{1 - l(1 - \tau_d)}, t = 0, \ldots, T - 1 \tag{20}
\]

\[
w_t^r = \frac{y_t}{1 - \tau_d} + \alpha l, t = 0, \ldots, Tr - 1 \tag{21}
\]
\[ w^*_t = \frac{y_t}{1 - \tau_d} (1 - h_t), \quad t = T_r, \cdots, T - 1 \]  

(22)

\[ c_t \geq 0, f_{kt} \geq 0, \quad 1 \geq \theta_{it} \geq 0, \quad i = 1, 2 \]  

(23)

At the terminal date \( T \), the value function take the value \( v_T \).

\[ v_T = \frac{\beta(1 - \beta^{H^1}) A_{H^1}^{1 - \gamma}}{(1 - \beta)(1 - \gamma)} \]  

(24)

We create a \( 5 \times 5 \times 5 \times 5 \times 8 \) grid for the state space such that \( s_{1t} \in [0.01, 1] \), \( s_{2t} \in [0.01, 1] \), \( p_{1t-1}^* \in [0.01, 1.2] \), \( p_{2t-1}^* \in [0.01, 1.2] \), and \( y_t \in [0.1, 8] \). We optimize over this state space for each of the 80 periods. We solve the dynamic optimization problem by using backward induction and linear interpolation. We keep the state space relatively small to economize on computation time and difficulty. But grid points are selected in such a way that allows us to obtain qualitative inferences about the location allocation decisions for most of the interesting cases.
References


Table 1: Table of Parameter Values for the Base Portfolio

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Base portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis-price ratio of asset 1 ($p_{1t-1}^*$)</td>
<td>1.0</td>
</tr>
<tr>
<td>Basis-price ratio of asset 2 ($p_{2t-1}^*$)</td>
<td>1.0</td>
</tr>
<tr>
<td>Prior holdings in equity 1 ($s_{1t}$)</td>
<td>0.4</td>
</tr>
<tr>
<td>Prior holdings in equity 2 ($s_{2t}$)</td>
<td>0.4</td>
</tr>
<tr>
<td>Mean return of equity 1 ($\mu_1$)</td>
<td>9%</td>
</tr>
<tr>
<td>Standard deviation of equity 1 ($\sigma_1$)</td>
<td>20%</td>
</tr>
<tr>
<td>Mean return of equity 2 ($\mu_2$)</td>
<td>13%</td>
</tr>
<tr>
<td>Standard deviation of equity 2 ($\sigma_2$)</td>
<td>30%</td>
</tr>
<tr>
<td>Correlation coefficient ($\rho$)</td>
<td>0</td>
</tr>
<tr>
<td>Bequest horizon ($H$)</td>
<td>20</td>
</tr>
<tr>
<td>Retirement wealth ratio ($y$)</td>
<td>0.4</td>
</tr>
<tr>
<td>Interest rate ($r$)</td>
<td>6%</td>
</tr>
<tr>
<td>Inflation rate ($i$)</td>
<td>3.5%</td>
</tr>
<tr>
<td>Ordinary income tax rate ($\tau_d$)</td>
<td>36%</td>
</tr>
<tr>
<td>Capital gains tax rate ($\tau_g$)</td>
<td>20%</td>
</tr>
<tr>
<td>Risk aversion parameter ($\gamma$)</td>
<td>3</td>
</tr>
<tr>
<td>Fraction of total wealth earned as labor income ($l$)</td>
<td>15%</td>
</tr>
<tr>
<td>Fraction of labor income contributed to the TDRA ($\alpha$)</td>
<td>20%</td>
</tr>
</tbody>
</table>
This table shows the frequencies of various asset holdings, and their magnitudes in different accounts over the lifetime of the investor. The two bottom rows show the mean wealth allocations in these accounts. This table is based on the values of the base portfolio parameters. The investor is allowed to borrow up to 100% of her TBA wealth. The Magnitude column indicates the fraction of the wealth within a particular account allocated to a particular asset class or consumption. And the rest of the columns contain the frequencies of holdings of assets of particular asset class in different accounts over her lifetime. The bottom rows show how much of the wealth in each account is allocated for different asset classes and consumption.

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>TBA</th>
<th>TDRA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Consumption</td>
<td>Equity</td>
</tr>
<tr>
<td>&lt; −90%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&lt; −40%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&lt; −10%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&lt; 0%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&gt; 0%</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>&gt; 5%</td>
<td>57.50</td>
<td>100</td>
</tr>
<tr>
<td>&gt; 10%</td>
<td>22.50</td>
<td>100</td>
</tr>
<tr>
<td>&gt; 50%</td>
<td>0.00</td>
<td>100</td>
</tr>
<tr>
<td>Mean wealth allocations (Working age)</td>
<td>8.75%</td>
<td>92.73%</td>
</tr>
<tr>
<td>Mean wealth allocations (Retirement age)</td>
<td>3.92%</td>
<td>97.56%</td>
</tr>
</tbody>
</table>

Table 2: Asset Holding Frequency and Wealth Allocations for an Investor with Low Retirement Wealth Ratio ($y = .1$)
<table>
<thead>
<tr>
<th>Magnitude</th>
<th>TBA</th>
<th>TDRA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Consumption</td>
<td>Equity</td>
</tr>
<tr>
<td>&lt; −90%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&lt; −40%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&lt; −10%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&lt; 0%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>&gt; 0%</td>
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<td>100</td>
</tr>
<tr>
<td>&gt; 5%</td>
<td>60</td>
<td>100</td>
</tr>
<tr>
<td>&gt; 10%</td>
<td>32.50</td>
<td>100</td>
</tr>
<tr>
<td>&gt; 50%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Mean wealth allocations (Working age)</td>
<td>9.30%</td>
<td>28.77%</td>
</tr>
<tr>
<td>Mean wealth allocations (Retirement age)</td>
<td>4.14%</td>
<td>42.09%</td>
</tr>
</tbody>
</table>

This table shows the frequency of various asset holdings, and their magnitudes in different accounts over the lifetime of the investor. The two bottom rows show the mean wealth allocations in these accounts. This table is based on the values of the base portfolio parameters. The investor is allowed to borrow up to 100% of her TBA wealth. The Magnitude column indicates the fraction of the wealth within a particular account allocated to a particular asset class or consumption. And the rest of the columns contain the frequencies of holdings of assets of particular asset class in different accounts over her lifetime. The bottom rows show how much of the wealth in each account is allocated for different asset classes and consumption.
This table presents the average percentage changes in the certainty equivalent consumption level (during the working years) due to the changes in the retirement contribution limit. The changes are measured by measuring the changes in the certainty equivalent consumption due to the changes in the retirement contribution level parameter ($\alpha$). The retirement wealth ratios ($y$) are listed in the first column. The second column lists changes in the certainty equivalent consumption due to the change in the contribution level from 5% to 20% when borrowing is not allowed, and the third column lists changes in the certainty equivalent consumption due to the change in the contribution level from 5% to 20% when the investor can borrow up to 100% of her TBA wealth. Similarly, columns four and five list the changes in certainty equivalent consumption due to the changes in the contribution level from 20% to 30%, with and without the borrowing constraint, respectively.
Parameter values are of the base portfolio. The investor is not allowed to borrow. Panel (a) and (b) show the wealth allocations of an investor with low retirement wealth ratio \((y = 0.1)\) within the TBA and the TDRA. Panel (c) and (d) show the wealth allocations of an investor with high retirement wealth ratio \((y = 0.8)\) within the TBA and the TDRA. In panel (a) and (c) lines referred to as Total show the total allocations for consumption, bonds, and equities in the TBA.
Figure 2: Asset Allocation and Location Under Borrowing and Short Sale Constraints

Parameter values are of the base portfolio. The investor faces both borrowing and short sale constraints. (a) Combined equity holding in the TBA across time and retirement wealth ratio. (b) Combined equity holding in the TDRA across time and retirement wealth ratio. (c) Bond holding in the TBA across time and retirement wealth ratio. (d) Bond holding in the TDRA across time and retirement wealth ratio.
Parameter values are of the base portfolio. The investor is allowed to borrow up to 100% of her TBA wealth. (a) Combined equity holding in the TBA across time and retirement wealth ratio. (b) Combined equity holding in the TDRA across time and retirement wealth ratio. (c) Bond holding in the TBA across time and retirement wealth ratio. (d) Bond holding in the TDRA across time and retirement wealth ratio.
Parameter values are of the base portfolio. The plots present the impact of correlation structure on the level of equity holding in the TDRA. Panel (a) and (b) present the cases when borrowing is not allowed, and panel (c) and (d) present the cases when the investor is allowed to borrow up to 100% of her TBA wealth. Panel (a) presents three curves indicating the levels of equity holding in the TDRA depending on correlation coefficients of –0.2, 0, and 0.2. Panel (b) presents three curves indicating the levels of equity holding in the TDRA depending on correlation coefficients of –0.4, 0, and 0.4.
Parameter values are of the base portfolio. (a) Relationship between bond holding in the TBA and Sharpe ratio of the second risky asset. A vector of standard deviations, $\sigma_2 \in \{25\%, 30\%, 35\%, 40\%, 45\%, 50\%, 55\%, 60\%, 65\%\}$, is used to generate the Sharpe ratios. Then the Sharpe ratios are plotted against the bond holding in the TBA when borrowing is not allowed. In panel (b), (c), and (d), we consider an investor who is allowed to borrow up to 100% of her TBA wealth. Panel (b) presents her consumption allocations under various bequest motives. Panel (c) presents her combined equity holding in the TBA for various bequest motives. Panel (d) presents her bond holding in the TBA for various bequest motives.
Figure 6: Interaction between Borrowing Restrictions and Retirement Wealth Ratio, and Optimal Size of the TDRA

Parameter values are of the base portfolio. The plots in panel (a) and (b) depict bond holdings in the TDRA for three levels of borrowing constraints given the retirement wealth ratio ($y$). Panel (a) and (b)presents the cases for $y = 0.4$ and $y = 0.7$, respectively. The curves represent the cases when no borrowing is allowed (0% borrowing), when the investor is allowed to borrow up to 25% of her TBA wealth, and when the investor is allowed to borrow up to 100% of her TBA wealth. Panel (c) and (d) present various certainty equivalent consumption levels due to the variations in the retirement wealth ratios ($y$). Here value of $y$ is a proxy for the size of the TDRA. Panel (c) presents the case when the investor is restricted from borrowing. Panel (d) presents the case when the investor is allowed to borrow upto 100% of her TBA wealth.
Parameter values are of the base portfolio. When the investor is allowed to borrow or borrowing constraint is relaxed, she is allowed to borrow up to 100% of her TBA wealth. (a) Consumption under various restrictions over time. C1 represents consumption under both short sale and borrowing constraints, C2 represents consumption with only short sale constraint, and C3 represents consumption without borrowing or short sale constraints. (b) Allocation for consumption with borrowing and short sale constraints across time and retirement wealth ratio. (c) Allocation for consumption when borrowing constraint is relaxed but short sale constraints is not relaxed. (d) Allocation for consumption for individuals with low \((y = .1)\) and high \((y = .8)\) retirement wealth ratio facing borrowing constraint.