The Asymptotically Ideal Model and the Estimation of Technical Progress*

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Abstract

In this paper we investigate productivity growth and technical progress bias in the United States, using the flexible functional forms modeling approach employed by Diewert and Wales (1992) and Kohli (1994). We specify a new cost function, based on the (globally flexible) Asymptotically Ideal Model, that allows for technical progress as in Mitchell and Primont (1991), and estimate the model by imposing global curvature, using methods suggested by Gallant and Golub (1984). Our estimates of productivity growth and technical progress biases indicate that productivity growth has been declining and that productivity growth has been imports augmenting and labor and capital reducing.

JEL classification: C22; F33.

Keywords: Flexible functional forms; Asymptotically Ideal Model; Regularity conditions.

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1 Introduction

This paper investigates productivity growth and technical progress bias in the United States, building on a large body of recent literature that takes a flexible functional forms modeling approach. In particular, we follow the interesting and attractive work by Diewert and Wales (1992) and Kohli (1994) and estimate input demand functions in the context of a new flexible functional form based on the Asymptotically Ideal Model (AIM), introduced by Barnett and Jonas (1983) and employed and explained in Barnett, Geweke, and Wolfe (1991). In doing so, we use United States data, over the period from 1960 to 2002.

Our approach specifies a cost function, based on the (globally flexible) Asymptotically Ideal Model, that allows for technical progress as in Mitchell and Primont (1991). Motivated by the widespread practice of ignoring the theoretical regularity conditions, we estimate the new model and impose the curvature conditions globally, using methods suggested by Gallant and Golub (1984). Our estimates of productivity growth and technical progress biases, within the framework of three inputs (imports, labor, and capital), indicate that productivity growth in the United States has been declining and that productivity growth has been imports augmenting and labor and capital reducing.

The paper is organized as follows. Section 2 briefly describes a new AIM cost function that allows for technical progress and the methods for calculating productivity growth, technical progress biases, and price elasticities. In Section 3 we discuss the data and in Section 4 we deal with relevant computational considerations. In Section 5 we estimate the model, assess the results in terms of their consistency with optimizing behavior, and explore the economic significance of the results. The final section concludes the paper.

2 The AIM (\(n\))-TP Cost Function

Technical progress is usually modeled by introducing time, \(t\), as a variable. In practice there are many possible ways of doing so, and in this paper we follow Mitchell and Primont (1991) and introduce \(t\) into the AIM cost function through input prices. We assume that the effective input price \(p^*_i\) depends on the nominal input price \(p_i\) and \(t\), as follows

\[
p^*_i = \varphi_i(p_i; t) = f_i^{-1}(f_i(p_i) - k_i(t))
\]

where \(f_i\) and \(k_i\) are increasing functions. We can think of many possible forms for \(f_i\) and \(k_i\), but here we assume that \(f_i(p_i) = \log p_i\) and \(k_i(t) = \lambda_i t\), so that

\[
p^*_i = \exp(\log p_i - \lambda_i t) = p_i e^{-\lambda_i t}
\]

where \(\lambda_i\) is the constant exponential rate of augmentation for input \(i\).
We assume that technology is constant returns to scale and following Barnett, Geweke, and Wolfe (1991) define the total cost function as

\[ C(y, p; t) = yP(p; t) \]  

where \( y \) denotes output and \( P(p; t) \) is an input price aggregator function (or ‘unit cost function’), defined as a multivariate Müntz-Szatz series expansion

\[ P_n(p; t) = \sum_{z \in A_n} a_z \prod_{j=1}^{2^n} p_{ij}^{2^{-n}} \]  

where \( n \) is the order of expansion, \( a_z \) the unknown parameters, \( N \) the number of production factors, and \( A_n = \{(i_1, i_2, \cdots, i_{2^n}) : i_1, i_2, \cdots, i_{2^n} \in \{1, 2, \ldots, N\}; i_1 \leq i_2 \leq \cdots \leq i_{2^n}\} \).

Combining equations (1), (2) and (3), we obtain a new total cost function that allows for technical progress,

\[ C_n(y, p^*; t) = y \sum_{z \in A_n} a_z \prod_{j=1}^{2^n} (p_{ij}^*)^{2^{-n}} = y \sum_{z \in A_n} a_z \prod_{j=1}^{2^n} (p_{ij} e^{-\lambda_i t})^{2^{-n}} = y \sum_{z \in A_n} a_z \prod_{j=1}^{2^n} p_{ij}^{2^{-n}} e^{c_i t}, \]  

where \( c_i = -\lambda_i 2^{-n} \). In (4), total cost is a function of input prices, output, and time. Moreover, \( C_n(y, p^*; t) \) satisfies linear homogeneity in prices. In what follows, we refer to this new AIM total cost function with technical progress as the ‘AIM(n)-TP cost function.’ It is to be noted that the AIM(n)-TP cost function has a total of

\[ N + \frac{1}{2^n!} \prod_{i=0}^{2^n-1} (N - i) \]

parameters.

Differentiating the log of the AIM(n)-TP cost function (4) with respect to \( t \) yields what Berndt and Wood (1982) refer to as the ‘rate of cost diminution relation,’ for each \( t \)

\[ \frac{\partial \ln C_n(y, p; t)}{\partial t} = \frac{1}{C} \frac{\partial C}{\partial t} = \frac{y}{C} \frac{\partial P_n(p; t)}{\partial t} \]

Under the assumption of constant returns to scale, this rate of cost diminution is the negative of total factor productivity (TFP) obtained from the dual production function — see, for
example, Ohta (1974). Technical progress takes place when \( TFP > 0 \), corresponding to an upward shift of the production frontier (or equivalently to an inward shift of the isoquant map).

Applying Shephard’s lemma to (4) yields the system of input demand functions

\[
x_i = \frac{\partial C_n(y, p; t)}{\partial p_i}, \quad i = 1, \cdots N
\]

Taking the derivative of each input demand equation with respect to time and dividing by the estimated demands,

\[
\tau_i = \frac{1}{x_i} \frac{\partial x_i}{\partial t}, \quad i = 1, \cdots N
\]

yields a measure (denoted by \( \tau_i \) above) of the effect of technical change for each factor input — see, for example, Diewert and Wales (1992) and Kohli (1994). If \( \tau_i > 0 \ (\tau_i < 0) \), then technical progress is input \( i \) augmenting (reducing), meaning that more (less) of the input is required due to the passing of time. Following Kohli (1994) and Fox and Diewert (1999), we define these \( \tau_i \)'s to be technical progress biases for each factor input. If

\[
TFP = -\tau_i, \quad i = 1, \cdots N
\]

then technical change is said to be unbiased (neutral), in the sense that all factor inputs are affected to the same degree. This corresponds to a ‘homothetic shift’ of the isoquants leaving the marginal rate of substitution between any two inputs (measured along a ray through the origin) unaffected by technical progress. If, however, equation (5) does not hold, then technical change is said to be biased; this corresponds to a ‘non-homothetic shift’ of the isoquants, meaning that the marginal rate of substitution between any two inputs is affected by technical progress.

Finally, price elasticities can be calculated from the estimated input demand functions as follows

\[
\eta_{ij} = \frac{\partial x_i(y, p; t)}{\partial p_j} \frac{p_j}{x_i(y, p; t)}
\]

for inputs \( i \) and \( j \), where \( i, j = 1, \cdots, N \).

In what follows, and in order to avoid computational difficulties in the large nonlinear parameter space, we assume that \( n = 1 \) and \( N = 3 \), so that we work with the AIM(1)-TP cost function and three inputs.

With \( n = 1 \) and \( N = 3 \), (4) can be written as

\[
C_{n=1}(y, p; t) = yP_{n=1}(p; t)
\]

\[
= y \left( a_1 p_1 e^{2c_1 t} + a_2 p_2 e^{2c_2 t} + a_3 p_3 e^{2c_3 t} + a_4 p_1^{1/2} p_2^{1/2} e^{(c_1+c_2)t} + a_5 p_1^{1/2} p_3^{1/2} e^{(c_1+c_3)t} + a_6 p_2^{1/2} p_3^{1/2} e^{(c_2+c_3)t} \right)
\]

(6)
where the \(a_i\)'s and \(c_i\)'s are unknown parameters. Differentiating equation (6) with respect to prices and using Shephard’s lemma yields the following system of factor demand functions

\[
x_1 = y \left( a_1 e^{2c_1 t} + \frac{1}{2} a_4 p_1^{-1/2} p_2^{-1/2} e^{(c_1+c_2)t} + \frac{1}{2} a_5 p_1^{-1/2} p_3^{-1/2} e^{(c_1+c_3)t} \right)
\]

(7)

\[
x_2 = y \left( a_2 e^{2c_2 t} + \frac{1}{2} a_4 p_1^{-1/2} p_2^{-1/2} e^{(c_1+c_2)t} + \frac{1}{2} a_6 p_2^{-1/2} p_3^{-1/2} e^{(c_2+c_3)t} \right)
\]

(8)

\[
x_3 = y \left( a_3 e^{2c_3 t} + \frac{1}{2} a_5 p_1^{-1/2} p_3^{-1/2} e^{(c_1+c_3)t} + \frac{1}{2} a_6 p_2^{-1/2} p_3^{-1/2} e^{(c_2+c_3)t} \right)
\]

(9)

Notice that the AIM(1)-TP model has 9 free parameters (that is, parameters estimated directly) — the AIM(2)-TP model has 18 parameters, the AIM(3)-TP model has 45 parameters, and the AIM(4)-TP model has 156 parameters!

3 Data

We use annual data for the United States over the period from 1960 to 2002. The data consists of prices and quantities for output (which is a Divisia index of domestic output and exports) and three variable inputs, imports, labor, and capital. Our treatment of imports as an input to the production sector is standard in this literature — see, for example, Fox and Diewert (1999). The price and quantity aggregates for (domestic) output are derived from three components — private consumption, government consumption, and investment. The price and quantity data for private and government consumption are from the OECD national accounts, whereas those for investment are constructed using the increase in stocks and gross fixed capital formation prices and quantities both taken from the OECD national accounts. Similarly, the price and quantity data for exports and imports are from the OECD national accounts.

Following Fox (1997), price and quantity aggregates for labor (our second input) are derived from two components: wage earners and self employed plus unpaid family workers. The price aggregate, \(p_2\), is calculated by dividing the compensation of employees by the number of wage earners, using OECD national accounts data for the former and Labor Force Statistics OECD data for the latter. The price of self employed plus unpaid family workers is taken to be 0.4 times the price of employment, \(p_2\). Hence, the compensation of self employed plus unpaid family workers is calculated as \(0.4 \times p_2 \times \text{(civilian employment – wage earners)}\), with the data for civilian employment (that is, total employment) taken from the OECD Labor Force Statistics. Regarding capital (our third input, \(x_3\)), its quantity is calculated as \(x_{3t+1} = (1 - \delta) x_{3t} + I_t\), with the starting value of \(x_3\) being approximated by \(I_1 / [1 - (1 - \delta) / g_I]\), where \(g_I\) is the average annual growth rate of investment over the sample period.
period and \( \delta \) the depreciation rate.\(^1\) The price of capital, \( p_3 \), is calculated as \((R + \delta)p_I\), where \( p_I \) is the price of investment, \( I \), and \( R \) is the ex post rate of return.

## 4 Computational Considerations

The input demands system (7)-(9) can be written (with an error term appended) as

\[
x_t = \psi(p_t, \theta) + \epsilon_t
\]  

(10)

In (10), \( x = (x_1, x_2, x_3)' \), \( \psi(p, \theta) = (\psi_1(p, \theta), \psi_2(p, \theta), \psi_3(p, \theta))' \), and \( \psi_i(p, \theta) \) is given by the right-hand side of each of (7)-(9).

As Gallant and Golub (1984, p. 298) put it,

“all statistical estimation procedures that are commonly used in econometric research can be formulated as an optimization problem of the following type [Burguete, Gallant and Souza (1982)]:

\[
\hat{\theta} \text{ minimizes } \varphi(\theta) \text{ over } \Theta
\]

with \( \varphi(\theta) \) twice continuously differentiable in \( \theta \).”

We follow Gallant and Golub (1984) and use Zellner’s (1962) seemingly unrelated regression method to estimate \( \theta \). Hence, \( \varphi(\theta) \) has the form

\[
\varphi(\theta) = \frac{1}{T} \epsilon' \epsilon = \frac{1}{T} \sum_{t=1}^{T} [x_t - \psi(p_t, \theta)]' \hat{\Sigma}^{-1} [x_t - \psi(p_t, \theta)]
\]  

(11)

where \( T \) is the number of observations and \( \hat{\Sigma} \) is an estimate of the variance-covariance matrix of (10). In minimizing (11), as in Serletis and Shahmoradi (2005), we use the TOMLAB/NPSOL tool box with MATLAB — see http://tomlab.biz/products/npsol. NPSOL uses a sequential quadratic programming algorithm and is suitable for both unconstrained and constrained optimization of smooth (that is, at least twice-continuously differentiable) nonlinear functions.

The cost function should be a quasi-concave function in input prices, \( p_i \) \((i = 1, \ldots, n)\). If the curvature condition is violated, we follow Gallant and Golub (1984) and impose the

\[
\frac{(1 - \delta)^T I_1}{g_I^T} + \frac{(1 - \delta)^{T-1} I_1}{g_I^{T-1}} + \frac{(1 - \delta)^{T-2} I_1}{g_I^{T-2}} + \cdots + I_1 \approx \frac{I_1}{1 - (1 - \delta)/g_I}.
\]

\(^1\)Assuming that (reproducible) capital will depreciate to zero within \( T \) years and that the average annual investment growth rate is \( g_I \) from (1960-\( T \)) to 1960, then (reproducible) capital in 1960 is calculated as

\[
\frac{(1 - \delta)^T I_1}{g_I^T} + \frac{(1 - \delta)^{T-1} I_1}{g_I^{T-1}} + \frac{(1 - \delta)^{T-2} I_1}{g_I^{T-2}} + \cdots + I_1 \approx \frac{I_1}{1 - (1 - \delta)/g_I}.
\]
curvature condition. In particular, a sufficient condition for quasi-concavity of \( C(y, p, \theta; t) \) is

\[
g(p, \theta) = \max_z \left \{ z' \nabla^2 C(y, p, \theta; t) : z' \nabla C(y, p, \theta; t) = 0, z'z = 1 \right \}
\]

where \( \nabla C(y, p, \theta; t) = (\partial / \partial p) C(y, p, \theta; t) \), \( \nabla^2 C(y, p, \theta; t) = (\partial^2 / \partial p \partial p') C(y, p, \theta; t) \), and \( g(p, \theta) \) is non-positive (i.e., zero or negative) when the concavity (curvature) constraint is satisfied and positive when it is violated. \( g(p, \theta) \) is referred to as the ‘constraint indicator.’ Hence, we impose curvature by modifying the optimization problem as follows

\[
\text{minimize } \varphi(\theta) \quad \text{subject to } \max_{z \in \Omega} g(p, \theta) \leq 0,
\]

where \( \Omega \) is a finite set with the finite number of elements \( p_i \) (i = 1, \cdots, n). Curvature can be imposed at some representative point in the data (that is, locally), over a region of data points, or at every data point in the sample (that is, globally).

## 5 Empirical Results

In panel A of Table 1 we present a summary of results in terms of parameter estimates, standard errors, and curvature violations when the model is estimated without the curvature condition imposed. Clearly, the model violates curvature at 3 observations. Because regularity hasn’t been attained (by luck), we follow the suggestions by Barnett (2002) and Barnett and Pasupathy (2003) and estimate the model by imposing curvature, using methods suggested by Gallant and Golub (1984) and recently applied by Serletis and Shahmoradi (2005). Using NPSOL we performed the computations and report the results in panel B of Table 1.

In Figure 1 we present the rate of productivity growth in the United States (over the period from 1960 to 2002), computed as the negative of the rate of cost diminution — all results in this section (including the technical progress biases and the elasticities) have been acquired using numerical differentiation. When technical progress occurs, the rate of cost diminution is negative and the rate of productivity growth (shown in Figure 1) is positive. From (the first line of) Table 2, the average productivity growth rate was 0.668% from 1960-1973, 0.477% from 1974-1992, and 0.395% from 1993-2002. This declining productivity growth result was also found by Fox and Diewert (1999) — using the normalized quadratic variable profit function, they report an average U.S. productivity growth rate of 0.96% from 1960-1973 and 0.45% from 1974-1992.

The technical progress biases in Figures 2-4 indicate that productivity growth has been imports augmenting and labor and capital reducing. The labor (and to a smaller extent capital) reducing aspects of technical progress confirm the perception of labor and capital being replaced by technical progress. From the last three lines of Table 2, the average technical progress bias for imports has been increasing from 2.029% from 1960-1973, to
2.249% from 1974-1992, and 2.322% from 1993-2002, suggesting that technical progress has been assisting imports to the United States. Similarly, the (negative) average technical progress biases for labor and capital have been increasing; in the case of labor, from -1.058 from 1960-1973 to -1.004 from 1993-2002, and in the case of capital from -0.539 from 1960-1973 to -0.360 from 1993-2002.

All own price elasticities in Table 3 appear reasonable, with that for labor being relatively stable over time. The imports price elasticity has fallen (in absolute value) from 0.412 in 1960-1973 to 0.349 in 1993-2002, suggesting that imports have become less sensitive to price changes in recent years and that improvements in the U.S. trade balance may require large movements in the value of the U.S. dollar. The price elasticity for capital has increased in absolute value from 0.321 in 1960-1973 to 0.344 in 1993-2002, suggesting that capital has become somewhat more price sensitive. Finally, the cross price elasticities are reported in Table 4. Keeping in mind that a negative sign indicates complementarity and a positive sign indicates substitutability, we note that all three inputs are substitutes (although the elasticity estimates are quite small).

6 Conclusion

We introduced a new globally flexible functional form that allows for technical progress — the AIM(n)-TP cost function — and estimated total factor productivity, technical progress biases, and related price elasticities, using annual data for the United States over the period from 1960 to 2002. Within the framework of three inputs (imports, labor, and capital), our estimates show that productivity growth in the United States has been declining, a result consistent with recent evidence reported by Fox and Diewert (1999). Moreover, our estimates of the technical progress biases indicate that productivity growth has been imports augmenting and labor and capital reducing.
References


### Table 1

**PARAMETER ESTIMATES**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>A. Unconstrained</th>
<th>B. Curvature constrained</th>
</tr>
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<tbody>
<tr>
<td>$a_1$</td>
<td>.0204</td>
<td>.0083</td>
</tr>
<tr>
<td>$a_2$</td>
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<td>.3582</td>
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<td>$a_3$</td>
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<td>$a_4$</td>
<td>-.2461</td>
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<td>$a_5$</td>
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<td>.4220</td>
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<td>$c_1$</td>
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<td>$c_3$</td>
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<td>Objective value</td>
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<td>.0142</td>
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<td>Curvature violations</td>
<td>3</td>
<td>0</td>
</tr>
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Note: Sample period, annual data: 1960-2002.
Figure 1. Productivity Growth Rate (%)
<table>
<thead>
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</thead>
<tbody>
<tr>
<td>TFP</td>
<td>.668 (.01)</td>
<td>.477 (.04)</td>
<td>.395 (.01)</td>
<td>.527 (.12)</td>
</tr>
<tr>
<td>Imports bias</td>
<td>2.029 (.02)</td>
<td>2.249 (.04)</td>
<td>2.322 (.01)</td>
<td>2.194 (.12)</td>
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<tr>
<td>Labor bias</td>
<td>-1.058 (.02)</td>
<td>-1.019 (.09)</td>
<td>-1.004 (.02)</td>
<td>-1.028 (.02)</td>
</tr>
<tr>
<td>Capital bias</td>
<td>-.539 (.01)</td>
<td>-.407 (.02)</td>
<td>-.360 (.05)</td>
<td>-.439 (.07)</td>
</tr>
</tbody>
</table>

Note: Sample period, annual data: 1960-2002. Numbers in parentheses are standard deviations.
Figure 2. Technical Progress Bias for Imports (%)
Figure 3. Technical Progress Bias for Labor (%)
Figure 4. Technical Progress Bias for Capital (%)

Productivity growth (%)
### TABLE 3

**Average Own-Price Elasticities**

<table>
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<tbody>
<tr>
<td>Imports</td>
<td>-.412 (.006)</td>
<td>-.365 (.009)</td>
<td>-.349 (.002)</td>
<td>-.377 (.026)</td>
</tr>
<tr>
<td>Labor</td>
<td>-.184 (.005)</td>
<td>-.182 (.002)</td>
<td>-.182 (.002)</td>
<td>-.183 (.003)</td>
</tr>
<tr>
<td>Capital</td>
<td>-.321 (.006)</td>
<td>-.339 (.002)</td>
<td>-.344 (.002)</td>
<td>-.334 (.010)</td>
</tr>
</tbody>
</table>

Note: Sample period, annual data: 1960-2002. Numbers in parentheses are standard deviations.
### Table 4

**Average Cross-Price Elasticities**

<table>
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<tbody>
<tr>
<td>$\eta_{12}$</td>
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<td>.121 (.003)</td>
<td>.117 (.001)</td>
<td>.123 (.005)</td>
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<tr>
<td>$\eta_{13}$</td>
<td>.281 (.009)</td>
<td>.244 (.006)</td>
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<td>$\eta_{21}$</td>
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<td>.022 (.001)</td>
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<td>$\eta_{23}$</td>
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<td>.160 (.002)</td>
<td>.157 (.002)</td>
<td>.163 (.006)</td>
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<tr>
<td>$\eta_{31}$</td>
<td>.046 (.003)</td>
<td>.073 (.005)</td>
<td>.083 (.001)</td>
<td>.067 (.015)</td>
</tr>
<tr>
<td>$\eta_{32}$</td>
<td>.275 (.003)</td>
<td>.265 (.003)</td>
<td>.261 (.002)</td>
<td>.267 (.006)</td>
</tr>
</tbody>
</table>

Note: Sample period, annual data: 1960-2002.
Numbers in parentheses are standard deviations.