Asymmetries in a Common Pool

Natural Resource Oligopoly

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Abstract

We consider two groups of firms harvesting a common pool resource and selling their production on the same output market. They therefore compete on both the input and the output markets. Representative firms of these two groups (of potentially different sizes) differ from one another by their marginal costs. We then have a group of low marginal cost firms – referred to as “big” firms – and a group high marginal cost firms – referred to as “small” firms. We derive explicit Markov perfect equilibrium strategies and examine the effects of marginal cost differential and group size asymmetries on the outcomes of the game. The equilibrium strategies of the firms are characterized by three intervals of stocks over which they adopt different exploitation behavior. When the resource stock is less than a certain threshold, there is no exploitation at all. Above that threshold and below a second threshold, the firms exploit the resource at rates that are linear and increasing in the resource stock. From this second threshold on, the firms produce at the constant harvest rates they would adopt under a static Cournot game. We find that the presence of asymmetries induces discontinuities in the strategy of the big firms and consequently in the aggregate harvest rate. We also find that the small low cost firms begin exploiting the resource and revert to their static Cournot production at threshold resource stocks that are higher when they are in presence of big firms than when they are the only type in the industry. As for the big low cost firms, they begin exploiting the resource at a higher resource stock in the asymmetric case than in the symmetric case when their proportion in the industry is above some threshold, and at a lower resource stock when their share is below that threshold. They begin producing at their static Cournot harvest rate at a lower resource stock in the asymmetric setting than in the symmetric setting. We also find that the equilibrium outcomes admit one or three steady states depending on the range of the asymmetries. Moreover any of these steady states can be reached by varying the asymmetries.
1 Introduction

Studies of the economic dynamics of common pool resource exploitation typically assume that the economic agents exploiting the resource are all identical. Yet, in many situations, the heterogeneity of the agents is an inescapable characteristic of the problem. Think for example of the case of fisheries, where it is common to find a number of big multinational fishing firms competing with many small local fishermen for the exploitation of a common fishing ground. These big firms have access to large scale technologies and consequently face considerably lower marginal costs than the small local fishermen. Similarly, aquifers are often shared by a few large capacity users — for instance big bottling firms — and many small capacity users. In such cases, it seems important to take into account the heterogeneity of the agents in order to properly characterize the equilibrium. It is the purpose of this paper to introduce some form of heterogeneity into a common pool resource model and to analyze the impact of this heterogeneity on the equilibrium outcome of the dynamic game being played by the agents.

More precisely, we consider the exploitation of a renewable resource stock by a finite number of two types of agents: a low marginal cost type, which we will call the “big firms” for short, and a high marginal cost type, which we will call the “small firms”. The total number of firms, which we will assume fixed, will thus be divided into two groups of firms, identical within groups but different across groups. The distribution of the fixed number of firms between the two groups will be allowed to vary. Thus both the cost differential between the representative agents from each group and the relative size of the groups will be important parameters. We will occasionally refer to the former as the “cost asymmetry” and to the latter as the “size asymmetry”.

The situations described above will be modelled as an oligopolistic differential game in which two groups of firms, identical within groups, have access to the same renewable natural resource pool, which they exploit in common. They then sell their harvest on the same output market.
We restrict attention to non cooperative equilibria in stationary Markov strategies. A stationary Markov strategy is a decision rule that is contingent only upon the current state of the game. In our context, it specifies the firm’s extraction rate as a function of the current stock of the resource.

A number of authors have analyzed the problem of the exploitation of a common pool resource in a differential game framework. Amongst them, Dockner et al. [2], Dockner and Sorger [3], Fischer and Mirman [4, 5], Gaudet and Lohoues [6], Levhari and Mirman [8], Plourde and Yeung [11], consider cases where the agents involved compete only for the exploitation of the resource, but do not compete on the output market. In those papers, the benefit functions of the agents depend on their own production only and not on that of their rivals. In this paper, the agents compete in the output market as well as in the exploitation of the resource, as in Benchekroun [1], Karp [7] and Mason and Polasky [10]. Those authors also assume benefit functions that depend not only on the agents’ own production, but also, through the market demand, on the production of their rivals. However, they assume identical agents when comes the time to derive equilibrium strategies. We will allow for heterogeneous agents.

Indeed, exploitation of a common pool natural resource may involve heterogeneous agents that incur different operating costs, depending on their intrinsic exploitation capability. In real life, examples of this kind of heterogeneity among agents abound. An example is the case of fisheries, where it is common to find one or a few “big” multinational fishing firms competing with many “small” local firms, or simply local fishermen exploiting the fishing grounds for subsistence. These multinationals which generally use bigger boats or more efficient techniques consequently incur lower marginal costs than the small firms or the fishermen. Another example is the existence of a small number of big water bottling firms exploiting an aquifer which is also used by many small firms or individuals.

Our model is more closely related to that of Benchekroun [1], in that, as in Benchekroun,
we consider a renewable natural resource characterized by a concave growth function which is approximated by two linear segments. As in Benchekroun also, we consider that all the firms sell the product of their harvest on the same output market, characterized by a downward sloping demand function. Benchekroun assumes two identical players exploiting the resource at zero marginal cost, and focuses on the effects on the equilibrium resource stock of a unilateral restriction of the exploitation of one firm and the corresponding adjustment in the rival’s exploitation. We assume a finite number of firms split into two groups and differentiated by their marginal costs. We focus on the effects of the cost asymmetry and the group size asymmetry on the individual strategies and the aggregate harvest rate, as well as on the dynamics of the resource stock and its steady states.

To do this, we derive a Markov perfect Nash equilibrium of our asymmetric model, compare the corresponding equilibrium strategies to those that arise in the symmetric model, and examine how the two types of asymmetries affect the steady states of the game. More precisely, we compare the situation where both types of firms coexist to the situations where either the small firms or the big firms are the only actors exploiting the resource. We also fully characterize, in terms of the parameters representing the cost asymmetry and the group size asymmetry, the types and the number of steady states obtained.

We find that, as in Benchekroun [1], the equilibrium strategies of the firms are characterized by three intervals of stocks over which they adopt different exploitation behavior. When the resource stock is less than a certain well identified threshold, there is no exploitation at all. Above that threshold and below a second threshold, the firms exploit the resource at rates that are linear and increasing in the resource stock. From this second threshold on, the firms produce at the constant harvest rates they would adopt under a static Cournot game. However, the presence of asymmetries induces discontinuities in the strategy of the big firms and consequently in the aggregate harvest rate. We also find that the small low cost firms begin exploiting the resource and revert to their
static Cournot production at threshold resource stocks that are higher when they are in presence of big firms than when they are the only type in the industry. As for the big low cost firms, they begin exploiting the resource at a higher resource stock in the asymmetric case than in the symmetric case when their proportion in the industry is above some threshold, and at a lower resource stock when their share is below that threshold. They begin producing at their static Cournot harvest rate at a lower resource stock in the asymmetric setting than in the symmetric one. We also find that the equilibrium outcomes admit one or three steady states depending on the range of the asymmetries. Moreover any of these steady states can be reached by varying the asymmetries.

The remainder of the paper is organized as follows. The model is presented and solved in Sections 2 and 3, respectively. In Section 4, we compare the outcomes of the asymmetric model to those obtained with the symmetric models when the firms are all identical, either big or small. In Section 5, we analyze the type and the number of steady states obtained given the range of values of the asymmetries. We conclude in Section 6.

2 The model

Consider a natural resource that is commonly owned and exploited by $n$ firms divided into two groups: a group of $n_b$ “big” firms and a group of $n_s$ “small” firms, with $n_s + n_b = n$. They are identical within a group but differ between groups by their (constant) marginal costs. The representative member from a given group $i$, $i = s, b$, has a marginal cost $w_i$. We will assume that $w_s \geq w_b$. Hence, the big firms have a marginal cost advantage over the small firms.

Let $x(t)$ denote the stock of the resource at time $t$ and $q_k(t)$ the rate of harvest of a given firm $k$, $k = 1, \ldots, n$. The inverse demand function for the output is

$$P(Q) = a - bQ,$$ (1)
where $a$ and $b$ are two positive constants. We assume that $a - w_i > 0, \ i = s, b$.

As in Benchekroun [1], we assume that the natural growth function of the resource takes the form:

$$g(x) = \begin{cases} \delta x & \text{for } x \leq x_{\text{max}}/2 \\ \delta (x_{\text{max}} - x) & \text{for } x > x_{\text{max}}/2 \end{cases},$$

(2)

where $\delta$ and $x_{\text{max}}$ are positive parameters reflecting the characteristics of the ecosystem. $\delta$ represents the intrinsic growth rate of the resource and $x_{\text{max}}$, the carrying capacity of the ecosystem. We assume that the intrinsic growth rate of the resource satisfies:

$$\delta > \frac{(n^2 + 1) \ r}{2},$$

(3)

where $r$ is the discount rate, assumed the same for all the firms. This condition is needed to guarantee convergence of the resource stock to strictly positive steady state levels, as will become clear in due course.

We restrict attention to equilibria in stationary Markov strategies. Stationary Markov strategies in this context are decision rules that specify a firm’s harvest rate as a function of the current resource stock: $q_k(t) = \phi_k(x(t))$. Firm $k$ takes the strategies of its $(n - 1)$ rivals as given in choosing its own decision rule, $q_k = \phi_k(x)$ in order to maximize the present value of its instantaneous profits:

$$J_k = \int_0^\infty e^{-rt} \left\{ \left[ P \left( q_k + \sum_{l \neq k} \phi_l(x) \right) - w_k \right] q_k \right\} dt$$

(4)

subject to

$$\dot{x} = g(x) - q_k - \sum_{l \neq k} \phi_l(x),$$

(5)
\( q_k \geq 0, \quad x(t) \geq 0. \tag{6} \)

An \( n \)-tuple of strategies \( (\phi_1(x),...,q_k,...\phi_n(x)) \) constitutes a Markov Perfect Nash Equilibrium if, for every possible initial condition \( x(0) = x_0 \), it simultaneously solves the above problem for \( k = 1,2,...,n \). Since the firms are identical within each group, it suffices to find a pair of Markov strategies \( (\phi_s(x),\phi_b(x)) \) which gives an \( n \)-tuple composed of \( n_s \) decision rules \( \phi_s(x) \) and \( n_b \) decision rules \( \phi_b(x) \) that satisfies this property.

### 3 Characterization of an equilibrium

In this section, we characterize a Markov perfect Nash equilibrium for this non-cooperative differential game. The following proposition provides such an equilibrium when not all the firms are of the same type.

**Proposition 1** Assume \( 0 < n_i < n \) and let \( \phi_i(x) \), \( i = s,b \), denote the following harvesting strategy:

\[
\phi_i(x) = \begin{cases} 
0 & \text{for } x \in [0,x_{1i}) \\
 f_i(x) \equiv \alpha (x-x_{1i}) & \text{for } x \in [x_{1i},x_{2i}) \\
 q_i^c & \text{for } x \in [x_{2i},x_{\max}] 
\end{cases} \tag{7}
\]

where, for \( i,j = s,b \), \( i \neq j \),

\[
\alpha = \frac{n+1}{n^2} \left( \delta - \frac{r}{2} \right), \tag{8}
\]

\[
q_i^c = \frac{1}{(n+1)b} \left[ a - w_i - n_j (w_i - w_j) \right], \tag{9}
\]

\[
x_{1i} = \frac{[\delta - (n^2 + 1)\frac{r}{2}]}{(n+1)^2b\delta (\delta - \frac{r}{2})} \left[ a - w_i + n_j (w_i - w_j) \frac{(2+n)(\delta - r)}{n(\delta - r)} \right], \tag{10}
\]


6
\[ x_{2i} = \frac{(n^2 + 1)}{(n + 1)^2 b \delta} \left[ a - w_i + n_j (w_i - w_j) \frac{[2 \delta - (n^2 + 1) r]}{n (n^2 + 1) (\delta - r)} \right]. \]  

(11)

The \( n \)-tuple \( (\phi_s, ..., \phi_s, \phi_b, ..., \phi_b) \) composed of \( n_s \) decision rules \( \phi_s(x) \) and \( n_b \) decision rules \( \phi_b(x) \) constitutes a Markov Perfect Nash Equilibrium.

**Proof.** See Appendix A.

Note that these strategies are such that there is no interval of resource stock over which only one type of firm produces. The level of stock above which both types of firms begin harvesting the resource is \( x_{1s} \). It is given by (10), with \( i = s \). When the resource stock is smaller than \( x_{1s} \), neither type produces.

Note also that, as in Benchekroun [1], \( q_i^c \) is the static Cournot equilibrium quantity of a firm of type \( i \). Hence, when the resource stock is sufficiently large (above a certain threshold), the Markov Perfect Nash Equilibrium consists in both types of firms simultaneously producing their static Cournot quantity. This threshold level of stock is the same for each type of firm and is given by \( x_{2s} \), obtained from (11) by setting \( i = s \).

In order to guarantee that \( x_{1b} > 0 \), \( x_{2b} \leq \frac{x_{\max}}{2} \) and \( x_{1b} < x_{2b} \), we will assume:

\[ \xi < \frac{n_s}{n} (w_s - w_b) < \frac{\delta - r}{[(2 + n) \delta - r]} (a - w_b). \]

(12)

where

\[ \xi = \max \left\{ \frac{(n^2 + 1)(\delta - r)}{2 \delta - (n^2 + 1)} \left[ (a - w_b) - \frac{(n + 1)^2 b \delta x_{\max}}{(n^2 + 1)} \right], \right. \]

\[ \left. \frac{(n + 1)}{\delta - \frac{(n + 1)}{2}} (w_s - w_b) - n (\delta - r) (a - w_b) \right\}. \]

(13)
As shown in Appendix B, we will then have:

\[ 0 < x_{1b} \leq x_{1s} < x_{2s} \leq x_{2b} < \frac{x_{\text{max}}}{2}. \] (14)

The resulting equilibrium strategies are illustrated in Figure 1, along with a possible corresponding aggregate production, \( \Phi(x) = n_s \phi_s(x) + n_b \phi_b(x) \).

A number of additional implications about the individual strategies can be drawn from Proposition 1. Firstly, both the strategies \( \phi_s(x) \) and \( \phi_b(x) \) are non decreasing functions of the resource stock, with the same non-negative slopes on each of the three distinct intervals over which they are defined. These slopes are given, for \( i = s, b \), by:

\[
\phi_i'(x) = \begin{cases} 
0 & \text{for } x \in [0, x_{1s}) \\
\alpha > 0 & \text{for } x \in [x_{1s}, x_{2s}) \\
0 & \text{for } x \in [x_{2s}, x_{\text{max}}] 
\end{cases}
\] (15)

They are independent of the marginal cost differential and of the distribution of both types of firms.

Secondly, the strategy \( \phi_s(x) \) is a continuous function of \( x \) over \([0, x_{\text{max}}]\), whereas \( \phi_b(x) \) exhibits jumps at both \( x_{1s} \) and \( x_{2s} \), unless all the firms in the industry are identical (i.e., \( w_s = w_b \)). Indeed, at \( x_{1s} \) we have:\footnote{We adopt the following notations: \( \phi_i(z^-) = \lim_{{x \to z^-, x < z}} \phi_i(x) \) and \( \phi_i(z^+) = \lim_{{x \to z^+, x > z}} \phi_i(x) \). The function \( \phi_i(x) \) is continuous at \( z \) if \( \phi_i(z^-) = \phi_i(z^+) \). Otherwise, \( \phi_i(x) \) exhibits a jump at \( x = z \).}

\[
\phi_b(x_{1s}^-) = 0 \quad \text{and} \quad \phi_b(x_{1s}^+) = \alpha (x_{1s} - x_{1b}),
\]
and hence

\[ \phi_b(x_{1s}^+) - \phi_b(x_{1s}^-) = \alpha(x_{1s} - x_{1b}) = \frac{[\delta - (n^2 + 1) \frac{r}{2}]}{n^2 b (\delta - r)} (w_s - w_b). \]  \hspace{1cm} (16) 

At \( x_{2s} \), we have

\[ \phi_b(x_{2s}^-) = \alpha(x_{2s} - x_{1b}) \quad \text{and} \quad \phi_b(x_{1s}^+) = q_b^c = \phi_b(x_{2b}) = \alpha(x_{2b} - x_{1b}), \]

and hence

\[ \phi_b(x_{2s}^+) - \phi_b(x_{2s}^-) = \alpha(x_{2b} - x_{2s}) = \frac{n - 1}{n^2 b} \frac{(\delta - r)}{(\delta - r)} (w_s - w_b). \]  \hspace{1cm} (17) 

The jumps observed in \( \phi_b(x) \) at \( x_{1s} \) and \( x_{2s} \) are proportional to the marginal cost differential. In both cases, the larger the cost differential, the larger the jump.

Thirdly, the difference in the harvest rate of the two types of firms, \( \phi_b(x) - \phi_s(x) \), is proportional to the cost differential, \( w_s - w_b \), and given by:

\[ \phi_b(x) - \phi_s(x) = \begin{cases} 
0 & \text{for } x \in [0, x_{1s}) \\
\frac{[\delta - (n^2 + 1) \frac{r}{2}]}{n^2 b (\delta - r)} (w_s - w_b) & \text{for } x \in [x_{1s}, x_{2s}) \\
\frac{1}{b} (w_s - w_b) & \text{for } x \in [x_{2s}, x_{\text{max}}] 
\end{cases}. \]  \hspace{1cm} (18) 

The big firms have a higher harvest rate as a consequence of their cost advantage.

Finally and not surprisingly, when \( w_s = w_b \) the strategies described in Proposition 1 reduce to those found in Benchekroun [1].
At the aggregate level, the overall harvest rate, $\Phi(x) = n_s\phi_s(x) + n_b\phi_b(x)$, is given by:

$$\Phi(x) = \begin{cases} 
0 & \text{for } x \in [0, x_{1s}) \\
F(x) \equiv n\alpha (x - \bar{x}_1) & \text{for } x \in [x_{1s}, x_{2s}) \\
Q^c & \text{for } x \in [x_{2s}, x_{max}] 
\end{cases}$$

where

$$\bar{x}_1 = \frac{n_s}{n}x_{1s} + \frac{n_b}{n}x_{1b} = \frac{2\delta - (n^2 + 1)r}{(n + 1)^2 b\delta (2\delta - r)} \left[ a - w_s + \frac{n_b}{n}(w_s - w_b) \right],$$

and

$$Q^c = \frac{n}{(n + 1)b} \left[ a - w_s + \frac{n_b}{n}(w_s - w_b) \right].$$

Note that when (3) holds, we also have that $\delta > (n + 1)\frac{r}{2}$ and therefore,

$$n\alpha - \delta = \frac{1}{n} \left[ \delta - (n + 1)\frac{r}{2} \right] > 0,$$

which means that the slope of $\Phi(x)$ is greater than that of $g(x)$ over the entire interval $[x_{1s}, x_{2s})$.

Like for the individual strategies, the slopes of $\Phi(x)$ over any of the three intervals over which it is defined are independent of the marginal cost differential and of the distribution of the firms between the two types. These slopes are 0, $n\alpha$, and 0 over $[0, x_{1s})$, $[x_{1s}, x_{2s})$ and $[x_{2s}, x_{max}]$, respectively. A possible representation of $\Phi(x)$ is depicted in Figure 1.\(^2\)

The equilibrium aggregate production exhibits jumps at $x_{1s}$ and $x_{2s}$, due to the jumps in the big firms' equilibrium strategy at these two particular resource stock levels. The size of those jumps

\(^2\)We come back later to a discussion of all the possible representations of $\Phi(x)$ when we discuss the possible steady states, in Section 5.
is given by:

\[
\Phi(x_{1s}^+) - \Phi(x_{1s}^-) = n_b \alpha (x_{1s} - x_{1b}) = \frac{\delta - (n^2 + 1) \frac{r}{n} }{n (\delta - r) b} \frac{n_b}{n} (w_s - w_b),
\]  

(23)

and

\[
\Phi(x_{2s}^+) - \Phi(x_{2s}^-) = n_b \alpha (x_{2b} - x_{2s}) = \frac{n - 1}{n} \frac{(\delta - \frac{r}{n})}{n (\delta - \frac{r}{n})} \frac{n_b}{n} (w_s - w_b).
\]  

(24)

At both \(x_{1s}\) and \(x_{2s}\), for any given \(0 < n_b < n\), the size of the jump is proportional to both the marginal cost differential, \(w_s - w_b\) and the proportion of big firms, \(n_b/n\). Thus discontinuities in the aggregate production will always be observed unless \(w_s = w_b\), as is the case in Benchekroun [1] (see Figure 2). For any given cost differential, the greater the number of big firms, the greater the jump in aggregate production.

When \(w_s = w_b\), there will always be either one or three steady states, as shown in Benchekroun [1]. This is also the case when \(w_s > w_b\). However, when \(w_s > w_b\), steady states may occur at the stock levels for which there is a jump in the aggregate harvest. The resource growth function \(g(x)\) and the aggregate harvest function \(\Phi(x)\) then cross at a point of discontinuity in \(\Phi(x)\). We will call these “irregular” steady states, to distinguish them from the “regular” steady states, for which \(\Phi(x)\) and \(g(x)\) cross within a continuous segment of \(\Phi(x)\).

Before analyzing in more detail the different steady-state configurations in Section 5, we first consider in the next section the effects of the distribution of firms between the two types on their individual equilibrium strategies and on the aggregate outcome.
4 Effects of the Distribution of Firms

In this section, we assume the marginal costs differential to be positive and given. We focus on the effect of having the two types of firms coexisting, rather than having all the firms of the same type.

To do this, we compare the equilibrium strategies derived in Proposition 1, where $0 < n_b < n$, to those that arise when all firms are of the same type, with either $n_b = n$ or $n_s = n$. We will use the superscript "o" to denote situations where all the firms are of the same type.

From Proposition 1, and in accord with Benchekroun [1], the individual strategies when all the firms are of the same type ($w_s = w_b$), are, for $i = s, b$:

$$
\phi_i^o(x) = \begin{cases} 
0 & \text{for } x \in [0, x_{1i}^o) \\
 f_i^o(x) \equiv \alpha (x - x_{1i}^o) & \text{for } x \in [x_{1i}^o, x_{2i}^o) \\
 q_i^{co} & \text{for } x \in [x_{2i}^o, x_{\text{max}}] 
\end{cases}
$$

where,

$$
\alpha = \frac{n + 1}{n^2} \left( \delta - \frac{r}{2} \right),
$$

$$
q_i^{co} = \frac{1}{(n + 1) b} [a - w_i],
$$

$$
x_{1i}^o = \frac{\left[ \delta - \frac{r}{2} \right]}{(n + 1)^2 b \delta (\delta - \frac{r}{2})} [a - w_i],
$$

$$
x_{2i}^o = \frac{\left( n^2 + 1 \right)}{(n + 1)^2 b \delta} [a - w_i].
$$

Setting, $w_s = w_b = 0$, $n = 2$, and $x_{\text{max}} = 1$, these strategies reduce exactly to those derived by Benchekroun.

When $n_i = n$, so that all the firms are identical, with marginal cost $w_i$, $x_{1i}^o$ is the level of the
resource stock beyond which the firms choose to harvest at a positive rate and \( x_{2i}^0 \) is that beyond which they choose their static-Cournot harvest rate.

We further assume that \( x_{2s}^0 \geq x_{1b}^0 \), which requires:

\[
w_s - w_b \leq \frac{n^2}{(n^2 + 1)} \left( \frac{\delta}{(\delta - \frac{\gamma}{2})} \right) (a - w_b).
\] (29)

Hence the marginal cost differential is assumed bounded from above.

To better understand the implications of the asymmetry in the distribution of firms between the two types of firms, we compare the equilibrium strategies in three cases, all with \( w_s > w_b \):

(i) Comparison of the equilibrium strategies when \( n_b = n \) and when \( n_s = n \);

(ii) Comparison of the equilibrium strategies of the small firms when \( n_s < n \) and when \( n_s = n \);

(iii) Comparison of the equilibrium strategies of the big firms when \( n_b < n \) and when \( n_b = n \).

We present those comparisons in the next three subsections and then discuss briefly the effect of the distribution of the types of firms on the aggregate harvest rate.

The detailed calculations required to make each of those comparisons are presented in Appendix C.

4.1 Comparing \( n_b = n \) and \( n_s = n \)

Contrary to the case where \( 0 < n_i < n, \ i = b, s \), if either \( n_s = n \) or \( n_b = n \), the equilibrium strategies for both types of firms, and therefore the aggregate harvesting rate, are everywhere continuous functions of the resource stock. Hence the jumps in the harvesting rates are due strictly to the simultaneous presence of both types of firms.

The equilibrium strategies and aggregate harvesting rates for \( n_s = n \) and \( n_b = n \) are juxtaposed in Figure 2.\(^3\) Note that \( x_{1s}^0 < x_{1b}^0 \). This means that if \( n_s = n \), so that there are only small firms,

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\(^3\)For the detailed calculations behind Figure 2, see Appendix C.
the threshold level of stock at which the firms begin exploiting the resource is lower than if \( n_b = n \), so that all the firms are big firms. Put differently, with identical firms, the lower the marginal cost, the higher the level of stock at which the firms begin exploiting the resource. In fact \( x_{0b} - x_{1s} \) is proportional to \( w_s - w_b \), as can be verified from equation (99) in Appendix C.

The decision not to harvest is a decision to invest in the resource by leaving it in place in order to generate growth. The reason why identical low cost firms (the big firms) tend to begin harvesting at a higher level of the stock than identical high cost firms (the small firms) is that, because of their lower cost of exploitation, each one of them values investing in the resource more than their high cost counterpart.

For the same reason, it is also the case that \( x_{0s}^0 < x_{2b}^0 \) (see equation (100) in Appendix C). This means that identical low cost firms \((n_b = n)\) will revert to their static Cournot strategy at a higher level of the stock than identical high cost firms \((n_s = n)\). As a result, there exists a stock level, which we denote \( x_0^0 \) in Figure 2, such that \( \phi_b^0 (x) < \phi_s^0 (x) \) for \( x_{1s} \leq x < x_0^0 \) and \( \phi_b^0 (x) > \phi_s^0 (x) \) for \( x > x_0^0 \). This is unlike the case with both types of firms coexisting \((0 < n_i < n)\). In that case, as can be seen from equation (18) and is illustrated in Figure 1, the harvesting rate of the big firms \( (\phi_b (x)) \) is higher than the harvesting rate of the small firms \( (\phi_s (x)) \) for all stock levels for which production is positive. From (25), we see that the stock level \( x_0^0 \) is given by \( x_0^0 = x_{1b}^0 + q_s^0 / \alpha \).

4.2 Comparing \( 0 < n_s < n \) and \( n_s = n \)

In Figure 3, we illustrate the harvest strategy of the small firms when some of their rivals are big low cost firms \((0 < n_s < n)\) to their strategy when they are the only firms operating in the industry \((n_s = n)\).

As derived in Appendix C, equation (104), and illustrated in Figure 3, we will have \( x_{1s} > x_{1s}^0 \). That is, if all \( n \) firms are small firms, each one of them begins harvesting the resource at a lower
stock level than if they are sharing the common resource pool with \( n - n_s \) big firms. Thus each high cost firm has a higher equilibrium valuation of investing in the common resource pool if some of its rivals are low cost firms than if all its rivals are also high cost firms. This is because each of them is aware that a positive harvest rate by the small firms immediately brings about competition from the big low cost firms for their share of the common resource pool.

Similarly, \( x_{2s} > x_{2s}^o \) (Equation (105), Appendix C), which means that the threshold level of stock at which a small firm reverts to its static Cournot output is lower if all its rivals are also small firms than if some of them are big low cost firms.

Furthermore, the static Cournot harvest rate of the high cost firm in the symmetric case \( (n_s = n) \), given by equation (26) with \( i = s \), will be higher than that in the asymmetric case \( (0 < n_s < n) \), given by equation (9) with \( i = s \).\(^4\) As a result, for all \( x > 0 \), \( \phi_s (x) \leq \phi_s^o (x) \), with \( \phi_s (x) < \phi_s^o (x) \) for all \( x > x_{1s}^o \). Thus the individual harvest rate of the small firm is lower in the presence of low cost firms amongst its rivals than it is otherwise, as illustrated in Figure 3.

### 4.3 Comparing \( 0 < n_b < n \) and \( n_b = n \)

In Figure 4, we display the strategy of the big firm in the asymmetric case \( (0 < n_b < n) \), where some of its rivals are small high cost firms, and in the symmetric case \( (n_b = n) \) where all its rivals are also big firms.

As already mentioned, in the asymmetric case, there is a jump in the harvest rate of the individual low cost firm at both \( x_{1s} \) and \( x_{2s} \). Such jumps do not occur in the symmetric case. Also, such jumps in the harvest rate never occurred in the case of the high cost firm.

Another important difference between the low and high cost firms is that, whereas \( x_{1s} > x_{1s}^o \),

\(^4\)To see why this is the case, assume \( n_s = n = 2 \). We have upward sloping reaction curves because demand and cost are linear. The symmetric equilibrium rate of production will be \( q_1^c = q_2^c = q^c \). Now assume that firm 2 is replaced by a lower cost firm. The reaction curve of 1’s rival then shifts up, resulting in an equilibrium rate of production for firm 1 of \( q_1^c > q^c \). Note that we are dealing with strategic substitutes here, since both demand and costs are linear.
so that the low cost firm begins exploiting the resource at a lower stock level in the symmetric case than in the asymmetric case, it is not necessarily the case that $x_{1b}^o < x_{1s}$. Indeed, from Appendix C, Equation (110), we get that:

$$x_{1s} - x_{1b}^o \geq 0 \iff \frac{n_b}{n} \geq \frac{1}{1 + \frac{(n+1)s}{\delta-r}} \equiv \rho^o. \quad (30)$$

This means that in the asymmetric case, if the proportion of big firms, $n_b/n$, is larger than a certain threshold, which we denote $\rho^o$, each big firm begins exploiting the resource at a stock level that is higher than if there were only big firms.\(^5\) Moreover, because the big firms start exploiting the resource at the same stock level as the small firms, and at a higher rate (because of their cost advantage), there results a jump in their equilibrium strategy. When the proportion of big firms is smaller than $\rho^o$, each begins exploiting the resource at a lower stock level than if they found themselves in the symmetric case, with only big firms as rivals, although they still begin exploiting at the same stock level as the small firms.

As for the threshold level of stock at which the big low cost firm reverts to its static Cournot output, we always have $x_{2s} < x_{2b}^o$ (Equation (118), Appendix C). That is, this threshold is always lower in the asymmetric case ($0 < n_b < n$) than in the symmetric case ($n_b = n$), whatever the proportion of big firms in the asymmetric case.

Moreover, the static Cournot harvest rate of the big low cost firm in the asymmetric case, given by equation (9) with $i = b$, will be greater than that in the symmetric case, given by equation (26) with $i = b$. As a result, for all $x > x_{1s}$, $\phi_b(x) > \phi_b^o(x)$. Thus, as illustrated in Figure 4, the individual harvest rate of the big firm is greater in the presence of small high cost firms amongst its rivals than it is otherwise, for all stock levels for which the big firm produces in the asymmetric

\(^5\)This is the case depicted in Figure 4.
case.

4.4 The Aggregate Harvest Rates

From (21) and (26), and using the fact that $a - w_b = (a - w_s) + (w_s - w_b)$, it is easy to show, as seen in Figure 5, that:

$$nq_s^{co} \leq Q^c \leq nq_b^{co}. \quad (31)$$

The inequality in (31) implies that when the resource is relatively abundant ($x \geq x_{2b}^o$), that is, on the portion of resource stock where all the firms produce at their static Cournot levels, the aggregate harvest rate is the largest when there are only big low cost firms in the industry, and the smallest when there are only small high cost firms. The aggregate harvest rate in the asymmetric case lies between the two.

For stock levels between $x_{1s}^o$ and $x_{2s}^o$, the comparison is not monotonic.

When $x_{1s}^o \leq x \leq x_{2s}^o$, the direction of the inequalities in (31) is reversed and we have

$$n\phi_b^o (x) \leq \Phi (x) \leq n\phi_s^o (x). \quad (32)$$

That is, for stock levels between $x_{1b}^o$ and $x_{2s}^o$, the aggregate harvest rate is the largest with the identical small high cost firms, and the smallest with the identical big low cost firms. In the asymmetric case the aggregate harvest rate again lies between the two.

For stock levels too small ($x \leq x_{1s}^o$), there is no harvest at all in any of the three cases examined.
5 Steady States

We now turn to the analysis of how both cost and size asymmetries affect the steady-state stocks. We focus only on the asymmetric model in which both types of firms coexist.\(^6\)

In order to do this, it is convenient to introduce the idea of a mean preserving marginal cost differential. Let \(\varepsilon\) denote this mean preserving marginal cost differential. Then:

\[
\varepsilon = w_s - w_b, \quad w_s = \bar{w} + \frac{\varepsilon}{2} \quad \text{and} \quad w_b = \bar{w} - \frac{\varepsilon}{2},
\]

(33)

where \(\bar{w}\) is the mean marginal cost \((\bar{w} = (w_s + w_b)/2)\), assumed constant.

Also, let \(\rho = n_b/n\) be the proportion of big firms in the industry.

Using these notations, individual \((\phi_s\) and \(\phi_b)\) and aggregate \((\Phi)\) equilibrium strategies can be rewritten as:

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Intervals</th>
<th>([0, x_{1s}))</th>
<th>([x_{1s}, x_{2s}))</th>
<th>([x_{2s}, x_{\text{max}}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi_s(x))</td>
<td>0</td>
<td>(f_s(x)) ≡ (\alpha(x - x_{1s}))</td>
<td>(q_s^c)</td>
<td></td>
</tr>
<tr>
<td>(\phi_b(x))</td>
<td>0</td>
<td>(f_b(x)) ≡ (\alpha(x - x_{1b}))</td>
<td>(q_b^c)</td>
<td></td>
</tr>
<tr>
<td>(\Phi(x))</td>
<td>0</td>
<td>(F(x)) ≡ (n\alpha(x - \bar{x}_1))</td>
<td>(Q^c)</td>
<td></td>
</tr>
</tbody>
</table>

where

\[
x_{1s} = \frac{[\delta - (n^2 + 1) \frac{r}{2}]}{(n + 1)^2 b \delta (\delta - \frac{r}{2})} \left\{ a - \bar{w} + \left[ \frac{(2 + n) \delta - r}{(\delta - r) \rho - \frac{1}{2}} \varepsilon \right] \right\},
\]

(34)

\[
x_{1b} = \frac{[\delta - (n^2 + 1) \frac{r}{2}]}{(n + 1)^2 b \delta (\delta - \frac{r}{2})} \left\{ a - \bar{w} + \left[ \frac{1}{2} - \frac{(2 + n) \delta - r}{(\delta - r) (1 - \rho)} \varepsilon \right] \right\},
\]

(35)

\(^6\)See Benchekroun [1] for the analysis of the existence and the number of steady states in the symmetric case, given the value of the intrinsic growth rate.
\[ \bar{x}_1 = (1 - \rho) x_{1s} + \rho x_{1b} = \frac{[\delta - (n^2 + 1) \frac{r}{n}]}{(n + 1)^2 b \delta (\delta - \frac{r}{n})} \left[ a - \bar{w} + \left( \rho - \frac{1}{2} \right) \varepsilon \right], \]  
(36)

\[ q_s^c = \frac{n}{(n + 1) b} \left\{ \frac{a - \bar{w}}{n} - \left( \frac{1}{2n} + \rho \right) \varepsilon \right\}, \]  
(37)

\[ q_b^c = \frac{n}{(n + 1) b} \left\{ \frac{a - \bar{w}}{n} + \left[ \frac{1}{2n} + (1 - \rho) \right] \varepsilon \right\}, \]  
(38)

\[ Q_c^c = \frac{n}{(n + 1) b} \left[ a - \bar{w} + \left( \rho - \frac{1}{2} \right) \varepsilon \right], \]  
(39)

\[ x_{2s} = \frac{(n^2 + 1)}{(n + 1)^2 b \delta} \left\{ a - \bar{w} + \left[ \frac{2\delta - (n^2 + 1) r}{(n^2 + 1) (\delta - r)} \rho - \frac{1}{2} \right] \varepsilon \right\}, \]  
(40)

and

\[ x_{2b} = \frac{(n^2 + 1)}{(n + 1)^2 b \delta} \left\{ a - \bar{w} + \left[ \frac{1}{2} - \frac{2\delta - (n^2 + 1) r}{(n^2 + 1) (\delta - r)} (1 - \rho) \right] \varepsilon \right\}. \]  
(41)

From equations (34) to (41), and as was illustrated in Section 4, we can see both size and cost asymmetries have an impact not only on the individual and aggregate amounts of resource extracted at equilibrium, but also on the level of stock at which the firms begin harvesting the resource, and the stock level at which they begin harvesting the resource at their respective Cournot quantities. Now, we will examine the impact of these asymmetries on the steady states.

### 5.1 Type and Number of Steady States

We analyze the number and the type of steady states, given the range of values of both asymmetries, focusing on the corresponding steady-state stocks. As in Benchekroun [1], we find that there are either one or three steady-state stocks. More precisely, we always have one steady-state stock (we denote by \( x^* \)) over the interval \([x_{1s}, x_{2s}]\), and either two steady states (\( x^{**} \) and \( x^{***} \)) or no steady

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\(^7\)In some particular cases we will have two steady state stocks.
state over the interval \([x_{2s}, x_{\text{max}}]\).

For ease of exposition, we will refer to steady states at which there is a jump in the aggregate harvest rate as “irregular” steady states, and those at which there is no jump as “regular” steady states. For irregular steady-state stocks, we do not have \(\Phi(x) - g(x) = 0\) as this would be the case at a regular steady-state stock. Over the interval \([x_{1s}, x_{2s}]\) the steady-state stock can be either regular (when \(x^* \in (x_{1s}, x_{2s})\)) or irregular (when \(x^* = x_{1s}\) or \(x^* = x_{2s}\)). The steady states over the interval \([x_{2s}, x_{\text{max}}]\), when they exist, are both regular. All these possibilities are depicted in Figures 6 to 11.

In what follows, we define the conditions under which we have regular or irregular steady states, and when we have one or three steady states.

Firstly, let us define the stock level \(\tilde{x}\) which is such that \(F(\tilde{x}) = g(\tilde{x})\). The stock level \(\tilde{x}\) corresponds to the intersection of the ascending part of \(g(x)\) with the increasing segment of \(\Phi(x)\) (Recall that \(\Phi(x)\) coincides with the function \(F(x)\) over the interval \([x_{1s}, x_{2s}]\)). We have

\[
\tilde{x} = \frac{n\alpha}{(n\alpha - \delta)} \bar{x}_1. \tag{42}
\]

Secondly, let us denote by \(\Delta_1, \Delta_2\) and \(\Delta_3\), the following terms:

\[
\Delta_1 = \tilde{x} - x_{1s}, \quad \Delta_2 = \tilde{x} - x_{2s} \quad \text{and} \quad \Delta_3 = g(x_{\text{max}}/2) - Q^e. \tag{43}
\]

Over the interval \([0, x_{1s}]\), there is no possible steady state since there is no production.

Over the interval \([x_{1s}, x_{2s}]\), there is a unique steady-state stock \(x^*\) which is such that:

- \(x^* = x_{1s}\) if \(\Delta_1 \leq 0\), that is, we have an irregular steady-state stock at \(x_{1s}\);
- \(x^* = x_{2s}\) if \(\Delta_2 \geq 0\), meaning that we have an irregular steady-state stock at \(x_{2s}\);
• $x^* = \tilde{x} \in (x_{1s}, x_{2s})$, if $\Delta_1 > 0$ and $\Delta_2 < 0$, which means that we have a regular steady-state stock at $\tilde{x}$.

Over the interval $[x_{2s}, x_{\text{max}}]$, we have either two regular steady states at $x^{**}$ and $x^{***}$ or no steady state at all.

• When $\Delta_3 < 0$, we have no steady state.

• When $\Delta_3 \geq 0$, we have two regular steady states at $x^{**}$ and $x^{***}$ defined as solutions of the equation $g(x) - Q^c = 0$.\(^8\) Then, we have

$$x^{**} = \frac{Q^c}{\delta}, \quad (44)$$

and

$$x^{***} = x_{\text{max}} - \frac{Q^c}{\delta}. \quad (45)$$

5.1.1 Regular or Irregular Steady State at $x^*$?

We have a regular steady-state stock at $x^*$ when $x_{1s} < \tilde{x} < x_{2s}$ (i.e. when $\Delta_1 > 0$ and $\Delta_2 < 0$) and an irregular steady-state stock otherwise. In particular, we have an irregular steady-state stock at $x_{1s}$ (i.e. $x^* = x_{1s}$) when

$$\Delta_1 \leq 0 \iff \varepsilon \geq \frac{a - \bar{w}}{\tau + \tau \rho} \equiv E(\rho), \quad (46)$$

\(^8\)When $\Delta_3 = 0$, we have a particular case in which $Q^c = \delta x_{\text{max}}/2$, and then $x^{**} = x^{***} = x_{\text{max}}/2$. In such a case, we have one steady state stock over the interval $[x_{2s}, x_{\text{max}}]$, and thus two steady state stocks overall.
and an irregular steady-state stock at $x_{2s}$ (i.e. $x^* = x_{2s}$) when

$$\Delta_2 \geq 0 \iff \varepsilon \geq \frac{a - \bar{w}}{\bar{x} + \tau \rho} = E(\rho),$$

where

$$\tau = \frac{\delta - (n^2 + 1) \frac{n}{\rho}}{n(\delta - r)} > 0.$$

Note that both irregular steady state cases are delimited by the same locus defined by equation $\varepsilon = E(\rho)$ (see Figures 12 and 13). Above that curve, the corresponding values of $\rho$ and $\varepsilon$ lead to one or the other of these two irregular steady-state stocks. In other words, above (and all along) this curve we have an irregular steady state (either $x_{1s}$ or $x_{2s}$) and below, we have a regular steady state. See Appendix D for the derivations leading to (46) and (47).

### 5.1.2 One or Three Steady States?

Since we always have one steady-state stock over the interval $[x_{1s}, x_{2s}]$, to verify whether we have one or three steady-state stocks is equivalent to verifying for what range of values of $\rho$ and $\varepsilon$ we have either no steady state or two steady states, over the interval $[x_{2s}, x_{\max}]$. This in turn is equivalent to verifying whether $\Delta_3 < 0$ or $\Delta_3 \geq 0$. In particular, the case of $\Delta_3 = 0$ gives an equation $\varepsilon = E_3(\rho)$ which delimits the $(\rho, \varepsilon)$-space for these two cases (see Figures 12 and 13 for the graphs and Appendix D for the derivations).

To be more specific, we have that

$$\Delta_3 \geq 0 \iff \varepsilon \leq E_3(\rho) = \frac{(n + 1) b}{n} \left[ g \left( \frac{x_{\max}}{2} \right) - Q_{1/2}^\rho \right] \frac{1}{\left( \rho - \frac{1}{2} \right)},$$

(49)
where

\[ Q^c_{1/2} = \frac{n(a - \bar{w})}{(n + 1)b}. \]  

(50)

Note that \( Q^c_{1/2} \) is the static Cournot aggregate production when both types firms are either in equal number (i.e. \( \rho = \frac{1}{2} \)) or when all firms are identical like in Benchekroun [1], with a marginal cost equal to \( \bar{w} \) (i.e. \( \varepsilon = 0 \) and \( w_i = \bar{w} \)). \( Q^c \) is greater than \( Q^c_{1/2} \) when the big firms outnumber the small firms (i.e. \( \rho > 1/2 \)).

Equation (49) implies that, in the (\( \rho, \varepsilon \))-space, below the locus defined by equation \( \varepsilon = E_3(\rho) \), our model admits three steady-state stocks and above it, only one steady-state stock. However, the shape of the curve corresponding to equation \( \varepsilon = E_3(\rho) \) depends on the sign of \( h_g(\frac{x_{\text{max}}}{2}) - Q^c_{1/2} \).

This sign in turn depends on how the parameter \( \delta \) is compared to a particular value \( \delta^o \) that only depends on parameters relating to the demand and growth functions but not on the asymmetries present in the model. Indeed, we have that

\[ \left[ g\left(\frac{x_{\text{max}}}{2}\right) - Q^c_{1/2} \right] \geq 0 \iff \delta \geq \frac{2n(a - \bar{w})x_{\text{max}}}{(n + 1)b} = \delta^o. \]  

(51)

5.2 Asymmetries and Steady States

Using (51), we can distinguish two cases: the case where \( \delta > \delta^o \) and that where \( \delta < \delta^o \). When the intrinsic growth rate \( \delta \) of the resource is larger than the particular value \( \delta^o \) defined in (51), we have \( \left[ g\left(\frac{x_{\text{max}}}{2}\right) - Q^c_{1/2} \right] > 0 \) and the corresponding graph is presented in Figure 12. The case \( \delta < \delta^o \) which implies that \( \left[ g\left(\frac{x_{\text{max}}}{2}\right) - Q^c_{1/2} \right] < 0 \) is depicted in Figure 13.\(^9\)

\(^9\)In fact, to be more rigorous, the two curves presented in Figures 12 and 13 must be completed by two other curves that take into account the assumptions in (12) which impose \( x_{1b} > 0 \) and \( x_{2b} < \frac{x_{\text{max}}}{2} \). However, we willingly
In the symmetric version of our model, (i.e. when all the firms are identical) the case $\delta > \delta^o$ would lead to three regular steady-state stocks, whereas when $\delta < \delta^o$ we would have only one regular steady state.

In our asymmetric model, however, in both cases, it will always be possible to get to all the combinations of steady states depicted in Figures 6 to 11. This is possible because of the presence of the asymmetries. Indeed, as can be seen in Figures 12 and 13, in both cases, the curves derived in the preceding subsection divide the $(\rho, \varepsilon)$-space in four quadrants:

- **Quadrant I**: This quadrant corresponds to the possibility presented in Figure 6. There are three regular steady-state stocks: $x^* = \tilde{x}$, $x^{**}$ and $x^{***}$, two of which are stable ($\tilde{x}$ and $x^{***}$). Which stable steady-state stock the game will lead to depends on the initial resource stock $x_0$. If $x_0 < x^{**}$, the resource stock will converge to $\tilde{x}$, and when $x_0 > x^{**}$ the resource stock will converge to $x^{***}$. In this last case, the strategies in which each type plays its Cournot quantity is sustainable and can be played indefinitely (at $x^{***}$).

- **Quadrant II**: When $(\rho, \varepsilon)$ falls in this quadrant, there is only one regular and stable steady state ($x^* = \tilde{x}$), as illustrated in Figure 7. In this case, the equilibrium resource stock converges to $\tilde{x}$.

- **Quadrant III**: This corresponds to the cases where there is one irregular steady-state stocks as in Figures 9 and 11. In Figure 9, the steady-state stock is $x_{2s}$, which is stable but irregular, because of the jump that occurs in the aggregate harvest rate at that steady-state stock. The equilibrium resource stock will always converge to $x_{2s}$, but the aggregate steady-state harvest rate will depend on the level of the initial stock. If $x_0 > x_{2s}$, the aggregate output will be the Cournot quantity, which the firms can extract indefinitely in a sustainable manner by all

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ignore these last two curves to focus on those partitionning the $(\rho, \varepsilon)$-space from conditions relating to the type and the number of steady states.
playing their individual Cournot quantities. However, if the $x_0 < x_{2s}$, the overall output is less than the aggregate Cournot quantity. In Figure 11, the steady-state stock is $x_{1s}$ which is a stable but irregular steady-state stock. In that figure, the equilibrium resource stock always converges to $x_{1s}$. However, for $x_0 < x_{1s}$, there will be no harvest.

- In Quadrant IV, we have the cases where there are three steady states, but with one irregular. These cases are those presented in Figures 8 and 10. In Figure 8, the steady-state stocks are: $x_{2s}$ (irregular), $x^{**}$ and $x^{***}$. The steady-state stocks $x_{2s}$ and $x^{***}$ are both stable and the steady state that will be reached by the resource stock depends on the initial stock. If $x_0 > x^{**}$, the resource stock will converge to $x^{***}$. However, if $x_0 < x^{**}$, the equilibrium resource stock will converge to the irregular steady-state stock $x_{2s}$. In Figure 10, the steady states stocks are: $x_{1s}$ (irregular), $x^{**}$ and $x^{***}$ where $x_{1s}$ and $x^{***}$ are both stable. The steady state to which the resource stock will converge depends on the level of the initial stock. If $x_0 > x^{**}$ the resource stock will converge to $x^{***}$. If $x_0 < x^{**}$, it converges to the (irregular) steady-state stock $x_{1s}$.

In each of the two cases (when $\delta > \delta^{o}$ or $\delta < \delta^{o}$), by varying either $\rho$ or $\varepsilon$ keeping the other constant, it is possible to move from one quadrant to another. For instance in Figure 12, for a given cost asymmetry (marginal cost differential), it is possible to move from Quadrant I to IV or III, or from I to II etc. only by modifying the size asymmetry (proportion of big firms). Similarly, controlling for the size asymmetry, it is possible, for instance, to move from Quadrant I to II or III, or from I to IV.

This last point can have policy implications. Since in our model, the differences in cost and size are exogenously fixed, these asymmetries can be used as policy instruments to reach a previously fixed goal in terms of the situations described in our four quadrants. We have shown with our model
that it is possible to reach any of these quadrants by controlling for both types of asymmetries, in any of the two cases $\delta > \delta^o$ or $\delta < \delta^o$, even though these are out of the control of a policy maker.

6 Conclusion

We have presented a model of oligopolistic exploitation of a common pool resource where we have two types of firms differing in their marginal costs, with potentially different numbers of firms in each group. We have shown that both the marginal cost differential (cost asymmetry) and the distribution of firms (size asymmetry) do affect the outcomes of the game in important ways. This is seen by comparing the equilibrium outcomes obtained in the asymmetric case to those in the symmetric case, where all the firms are all identical, either big or small. We have also examined the implications for the steady states.

We have found that in our model, in addition to the “usual” strategic interactions due to the presence of other players (either of the same type or of different type), there are three other forces in action that lead to the behavior observed: (i) the common property effect which makes the big firms begin exploiting the resource at the same stock level as the small firms, which, in an identical-firm industry, would have started exploiting the resource at a lower stock level; (ii) the strategic effect that makes the small firm anticipate an earlier entry of the big firms, that will harvest at a higher rate, given their cost advantage; (iii) the demand effect, as a result of which the small firms begin their exploitation at a higher stock to avoid a decrease in the price of the output due to the higher harvest rate of the big firms. Compared to the symmetric case, the combination of these asymmetry-related forces finally results in: (a) discontinuities in the big firm harvest rate, and consequently in the aggregate outcome; (b) a forward shift in the beginning of the exploitation by the small firms and in the beginning of their static Cournot behavior as well. That is, in presence of big low cost firms, the small firms will begin exploiting the resource or start playing their Cournot
quantity at higher stock level than if they were the only type of firms in the industry; (c) a forward shift (when they are relatively more) or a backward shift (when relatively less) in the beginning of the exploitation by the big low cost firms and a forward shift in the beginning of their Cournot behavior when they are in presence of small high cost firms than when they are all identical. We have also found that the equilibrium outcomes admit one or three steady states depending on the range of values of the cost differential and the proportion of big firms. Moreover, any of these steady states can be reached by controlling the asymmetries. In such a case, our model can lead to interesting policy issues for a policy maker if it is possible to control the asymmetries.

Our analysis has assumed that the agents differ by their marginal costs and that the number of each type of firms is exogenously given. This leaves room for further research involving both other forms of asymmetry and endogenously determined numbers of firms.
7 APPENDIX

7.1 APPENDIX A: Proof of proposition 1

This proof is a constructive one. Let us consider the strategies $\phi_i(x)$, $i = s, b$, proposed in (7). For the vector $(\phi_s, ..., \phi_s, \phi_b, ..., \phi_b)$ to constitute a Markov perfect Nash equilibrium, we need to show that there exists two value functions $V_s(x)$ and $V_b(x)$, defined over $[0, x_{\text{max}}]$, such that, the guess $\phi_k(x)$ is solution of the following Hamilton-Jacobi-Bellman equation:

$$rV_k(x) = \max_{q_k} \left\{ \left[ a - w_k - b \left( q_k + \sum_{l \neq k} \phi_l(x) \right) \right] q_k + V_k'(x) \left[ g(x) - q_k - \sum_{l \neq k} \phi_l(x) \right] \right\}. \quad (52)$$

We start by checking for interior solutions, that is, for $x \in [x_{1s}, x_{2s})$ and $x \in [x_{2s}, x_{\text{max}}]$. Maximization of the right-hand side of (52) gives the following first order condition,

$$a - w_k - 2bq_k - b \sum_{l \neq k} \phi_l(x) - V_k'(x) = 0. \quad (53)$$

When considering the fact that there are two groups of firms and that at equilibrium, $q_i = \phi_i(x)$, from (53) we get the following system of equations, for $i, j = s, b$, $i \neq j$:

$$b(n_i + 1) \phi_i(x) + bn_j \phi_j(x) = a - w_i - V_i'(x), \quad (54)$$

from which we derive the following solution, for $i, j = s, b$, $i \neq j$:

$$\phi_i(x) = \frac{1}{b(n + 1)} \left\{ a - w_i - V_i' - n_j \left[ (V_i' - V_j') + (w_i - w_j) \right] \right\}. \quad (55)$$

If we designate by $\Phi(x) = \sum_k \phi_k(x)$ and by $\Lambda(x) = \sum_k V_k(x)$ the aggregate equilibrium harvest
rate and value function, respectively, we also have:

\[ b(n+1) \Phi(x) = n(a - w_i) + n_j (w_i - w_j) - \Lambda'(x). \]  

(56)

At equilibrium, the Hamilton-Jacobi-Bellman equation in (52) becomes, for \( i, j = s, b, i \neq j \):

\[ rV_i(x) = [a - w_i - b\Phi(x)] \phi_i(x) + V_i'(x) [g(x) - \Phi(x)] \]  

(57)

where

\[ \Phi(x) = n_i \phi_i(x) + n_j \phi_j(x). \]  

(58)

By substituting the \( \phi_i \)'s from (55) and (58) into (57), we get the following system of differential equations, for \( i, j = s, b, i \neq j \):

\[ (n+1)^2 b r V_i(x) = [(a - w_i) - n_j (w_i - w_j)]^2 \]

\[ + V_i'(x) \left\{ b(n+1)^2 g(x) - (n^2 + 2n_j) (a - w_i) - 2n_i n_j (w_i - w_j) \right\} \]

\[ + V_j'(x) \left\{ 2n_j [(a - w_i) - n_j (w_i - w_j)] \right\} \]

\[ + \left\{ n_i V_i'(x) + n_j V_j'(x) \right\}^2 \]  

(59)

To solve (59), when \( x \in [x_{1s}, x_{2s}) \) and \( x \in [x_{2s}, x_{\text{max}}] \), we use the “undetermined coefficients technique” to determine the value functions \( V_s(x) \) and \( V_b(x) \), for each of these two intervals over which the \( \phi_i \)'s are interior solutions.

- **For** \( x \in [x_{1s}, x_{2s}) \)
Let us start by the interval \([x_1, x_2]\), over which both strategies are strictly increasing functions.

To solve (59) for \(x \in [x_1, x_2]\), we use a “guess and verify” method, by guessing a quadratic form for the value functions, i.e. for \(i, j = s, b, i \neq j\):

\[
V_i(x) = \frac{A_i}{2} x^2 + B_i x + C_i, \quad (60)
\]

where \(A_s, A_b, B_s, B_b, C_s\) and \(C_b\) are parameters to be determined.

Replacing this guess in (59) gives, for \(i, j = s, b\):

\[
\begin{align*}
\left\{ b (n + 1)^2 A_i \left( \delta - \frac{r}{2} \right) + (n_i A_i + n_j A_j)^2 \right\} x^2 \\
+ \left\{ b (n + 1)^2 B_i \left( \delta - r \right) - A_i \left[ (n^2 + 1 + 2n_j) (a - w_i) + 2n_i n_j (w_i - w_j) \right] \\
+ 2n_j A_j \left[ (a - w_i) - n_j (w_i - w_j) \right] + 2 (n_i A_i + n_j A_j) \left( n_i B_i + n_j B_j \right) \right\} x \\
+ \left[ (a - w_i) - n_j (w_i - w_j) \right]^2 - B_i \left[ (n^2 + 1 + 2n_j) (a - w_i) + 2n_i n_j (w_i - w_j) \right] \\
+ 2n_j B_j \left[ (a - w_i) - n_j (w_i - w_j) \right] + (n_i B_i + n_j B_j)^2 - (n + 1)^2 brC_i \\
= 0
\end{align*}
\] (61)

Since (61) must hold for any \(x \in [x_1, x_2]\), this imposes all the coefficients of this second degree polynomial to be zero. Then, we get the following system of equations, for \(i, j = s, b\):

\[
b (n + 1)^2 A_i \left( \delta - \frac{r}{2} \right) + (n_i A_i + n_j A_j)^2 = 0
\] (62)
\[ b(n+1)^2 B_i (\delta - r) - A_i \left[ (n^2 + 1 + 2n_j) (a - w_i) + 2n_i n_j (w_i - w_j) \right] \]
\[ + 2n_j A_j \left[ (a - w_i) - n_j (w_i - w_j) \right] + 2 (n_i A_i + n_j A_j) (n_i B_i + n_j B_j) \]
\[ = 0 \]

\[ [(a - w_i) - n_j (w_i - w_j)]^2 - B_i \left[ (n^2 + 1 + 2n_j) (a - w_i) + 2n_i n_j (w_i - w_j) \right] \]
\[ + 2n_j B_j \left[ (a - w_i) - n_j (w_i - w_j) \right] + (n_i B_i + n_j B_j)^2 - (n + 1)^2 brC_i \]
\[ = 0 \]

from which we obtain the values of the unknowns \( A_i, B_i \) and \( C_i \) specified in (60). That is, for \( i, j = s, b, i \neq j \):

\[ A_i = A_j = A = -\frac{(n+1)^2 b}{n^2} \left( \frac{\delta - r}{2} \right) \]

\[ B_i = -\frac{A}{(n+1)^2 b \delta} \left[ (n^2 + 1) (a - w_i) + n_j (w_i - w_j) \left( \frac{2\delta - (n^2 + 1) r}{n (\delta - r)} \right) \right] \]

\[ = \frac{\left( \frac{\delta - r}{2} \right)}{n^2 \delta} \left[ (n^2 + 1) (a - w_i) + n_j (w_i - w_j) \frac{2\delta - (n^2 + 1) r}{n (\delta - r)} \right] \]

and

\[ C_i = \frac{1}{(n+1)^2 br} \left\{ [(a - w_i) - n_j (w_i - w_j)]^2 + (n_i B_i + n_j B_j)^2 \right. \]
\[ - (a - w_i) \left[ (n^2 + 1) B_i + 2n_j (B_i - B_j) \right] - 2n_j (w_i - w_j) (n_i B_i + n_j B_j) \]
Some helpful relations:

\[
B_i - B_j = A \frac{(n^2 - 1)}{(n + 1)^2 b} \frac{(\delta - r)}{(\delta - r)} (w_i - w_j) 
\]

(68)

\[
= -\frac{(n^2 - 1)}{n^2} \frac{(\delta - r)}{(\delta - r)} (w_i - w_j)
\]

\[
n_i B_i + n_j B_j = -A \frac{(n^2 + 1)}{(n + 1)^2 b} \left[ n (a - w_i) + n_j (w_i - w_j) \right]
\]

(69)

\[
= \frac{(n^2 + 1)}{n^2} \frac{(\delta - \frac{r}{2})}{\delta} \left[ n (a - w_i) + n_j (w_i - w_j) \right]
\]

\[
B_i - (a - w_i) = \frac{2\delta - (n^2 + 1) r}{2n^2 \delta} \left[ (a - w_i) + n_j (w_i - w_j) \frac{2\delta - r}{n (\delta - r)} \right]
\]

(70)

\[
(B_i - B_j) + (w_i - w_j) = (w_i - w_j) \frac{2\delta - (n^2 + 1) r}{2n^2 (\delta - r)}
\]

(71)

When the assumption in (3) holds and when \( w_s - w_b > 0 \), we have:

\[
A < 0, \quad B_b > B_s > 0
\]

(72)

For \( i, j = s, b, i \neq j \), let us define the functions \( W_i : [0, x_{\text{max}}] \rightarrow \mathbb{R} \) and \( f_i : [0, x_{\text{max}}] \rightarrow \mathbb{R} \) such that:

\[
W_i(x) \equiv \frac{A}{2} x^2 + B_i x + C_i
\]

(73)

and

\[
f_i(x) \equiv \frac{1}{b(n + 1)} \left[ -Ax + (a - w_i - B_i) - n_j (w_i - w_j + B_i - B_j) \right],
\]

(74)
and the stock levels

\[ x_{1i} = \frac{1}{-A} [B_i - (a - w_i) + n_j (B_i - B_j + w_i - w_j)] \]
\[ \equiv \frac{1}{-A} \left[ \frac{2\delta - (n^2 + 1) r}{2n^2 \delta} \right] [(a - w_i) + n_j (w_i - w_j)] \frac{(2 + n) \delta - r}{n (\delta - r)} \]

(75)

which are such that \( f_i(x_{1i}) = 0 \). In other terms,

\[ f_i(x) \equiv \frac{-A}{b(n + 1)} [x - x_{1i}] = \alpha (x - x_{1i}). \]

(76)

Note that

\[ x_{1i} - x_{1j} = \frac{(n + 1) \left[ 2\delta - (n^2 + 1) r \right]}{-A \frac{2n^2 (\delta - r)}{2n^2 (\delta - r)}} (w_i - w_j), \]

(77)

and

\[ f_i(x) - f_j(x) = \frac{A}{b(n + 1)} [x_{1i} - x_{1j}]. \]

(78)

The stock levels \( x_{2i} \) are defined such that \( W_i'(x_{2i}) = 0 \). We have

\[ x_{2i} = \frac{B_i}{-A} \]

(79)

from which we can get:

\[ x_{2i} - x_{2j} = \frac{B_i - B_j}{-A}. \]

(80)
It will be demonstrated in Appendix B that:

$$0 < x_{1b} \leq x_{1s} < x_{2s} \leq x_{2b} \leq \frac{x_{\text{max}}}{2},$$

(81)

and,

$$f_b(x) \geq f_s(x), \text{ for all } x \geq x_{1b}. \tag{82}$$

To sum up, for $x \in [x_{1s}, x_{2s})$, $V_b$ and $V_s$ coincide with $W_b$ and $W_s$, respectively and are continuously differentiable over $(x_{1s}, x_{2s})$. Then, by construction, for $x \in [x_{1s}, x_{2s})$, the strategies $\phi_b(x)$ and $\phi_s(x)$ are the restrictions of $f_b(x)$ and $f_s(x)$, respectively, over this interval.

- For $x \in [x_{2s}, x_{\text{max}}]$

For $x \geq x_{2s}$, the guessed strategies are interior solutions and are both constant functions:

$$\phi_s(x) = q_s^c, \quad \phi_b(x) = q_b^c, \quad \Phi(x) = n_s q_s^c + n_b q_b^c. \tag{83}$$

Over this interval, (57) becomes, for $i, j = s, b, i \neq j$:

$$rV_i(x) = \left[ a - w_i - b \left( n_i q_i^e + n_j q_j^e \right) \right] q_i^c + V'_i(x) \left[ \delta x - \left( n_i q_i^e + n_j q_j^e \right) \right]. \tag{84}$$

We guess that $V_i(x)$ is a constant function and get

$$V_i(x) = \frac{1}{r} \left[ a - w_i - b \left( n_i q_i^e + n_j q_j^e \right) \right] q_i^c.$$
The parameters \( q_i^c \) are obtained by using (55), from which we get,

\[
q_i^c = \frac{1}{b(n+1)} [a - w_i - n_j (w_i - w_j)],
\]

and consequently, for \( x \in [x_{2s}, x_{\text{max}}] \),

\[
V_i(x) = \frac{(a - w_i - n_j (w_i - w_j))^2}{(n+1)^2 \cdot br} \equiv Y_{2i}(x).
\]

To sum up, for \( x \in [x_{2s}, x_{\text{max}}] \), \( V_b \) and \( V_s \) coincide with \( Y_{2b} \) and \( Y_{2s} \) respectively and are continuously differentiable over \( (x_{2s}, x_{\text{max}}) \). Then, by construction, for \( x \in (x_{2s}, x_{\text{max}}) \), the strategies \( \phi_b(x) \) and \( \phi_s(x) \) are constant functions, and are equal to \( q_b^c \) and \( q_s^c \), respectively.

- **For** \( x \in [0, x_{1s}) \).

For \( x \in [0, x_{1s}) \), we have corner solutions. When, \( \phi_s(x) = \phi_b(x) = \Phi(x) = 0 \), (57) becomes

\[
rV_i(x) = \delta x V_i'(x).
\]

A solution of (87) that is continuous at \( x_{1s} \) is

\[
V_i(x) = W_i(x_{1s}) \left[ \frac{x}{x_{1s}} \right]^{\tau} \equiv Y_{1i}(x).
\]

### 7.2 Appendix B: Comparison of the \( x_{ki} \)'s

We show here that,

\[
0 < x_{1b} \leq x_{1s} < x_{2s} \leq x_{2b} \leq \frac{x_{\text{max}}}{2}.
\]
First, from (10) with \( i = b \), to have \( x_{1b} > 0 \), it must be true that:

\[
\frac{n_b}{n}(w_s - w_b) < \frac{\delta - r}{(2 + n)\delta - r}(a - w_b). \tag{90}
\]

Second, to have \( x_{2b} < x_{\text{max}}/2 \), from (11) with \( i = b \), we must verify that:

\[
\frac{n_s}{n}(w_s - w_b) > \frac{(n^2 + 1)(\delta - r)}{2\left[\delta - \frac{(n^2 + 1)}{2}\right]} \left[ (a - w_b) - \frac{(n + 1)^2 b \delta x_{\text{max}}}{(n^2 + 1)2} \right]. \tag{91}
\]

Third, from (77) with \( i = s \) and \( j = b \) we have

\[
x_{1s} - x_{1b} = \frac{[2\delta - (n^2 + 1)r]}{b(n + 1)(2\delta - r)(\delta - r)}(w_s - w_b) \geq 0, \tag{92}
\]

from which we deduce that \( x_{1s} \geq x_{1b} \).

Fourth, from (80) and (68) with \( i = s \) and \( j = b \), we can write

\[
x_{2s} - x_{2b} = -\frac{(n^2 - 1)}{(n + 1)^2 b(\delta - r)}(w_s - w_b) \leq 0, \tag{93}
\]

which shows that \( x_{2b} \geq x_{2s} \).

Fifth, from (75) and (79), using (71) with \( i = s \) and \( j = b \), we get

\[
x_{2s} - x_{1s} = \frac{2n^2}{(n + 1)^2 b(2\delta - r)} \left\{ (a - w_s) - \frac{n_b}{n} (w_s - w_b) \frac{[2\delta - (n^2 + 1)r]}{2n(\delta - r)} \right\}, \tag{94}
\]

which implies that, to have \( x_{2s} > x_{1s} \), it must be true that:

\[
\frac{n_b}{n}(w_s - w_b) < \frac{2n(\delta - r)}{[2\delta - (n^2 + 1)r]}(a - w_s), \tag{95}
\]

36
or written as in (91), it must be verified that

$$
\frac{n_s}{n} (w_s - w_b) > \frac{(n+1) \left[ \delta - \frac{(n+1)}{2} \right] (w_s - w_b) - n (\delta - r) (a - w_b)}{\delta - \frac{(n^2+1)}{2}}.
$$

(96)

To sum up, in order for (89) to be verified, it must be true that:

$$
\xi < \frac{n_s}{n} (w_s - w_b) < \frac{\delta - r}{(2 + n) \delta - r} (a - w_b),
$$

(97)

where

$$
\xi = \max \left\{ \frac{(n^2+1)(\delta - r)}{2 \left( \frac{(n^2+1)}{2} \delta - \frac{n(n+1)}{2} \right)} \left[ (a - w_b) - \frac{(n+1)^2 b \delta x_{\max}}{(n+1) \delta - \frac{(n^2+1)}{2}} \right] \right\}.
$$

(98)

### 7.3 APPENDIX C: Derivations for Section 4

Comparing $n_b = n$ and $n_s = n$

From (27), (28), and (26) we get:

$$
x_{1b}^o - x_{1s}^o = \frac{[2 \delta - (n^2 + 1) r]}{(n+1)^2 b \delta (2 \delta - r)} [w_s - w_b] > 0.
$$

(99)

$$
x_{2b}^o - x_{2s}^o = \frac{(n^2 + 1)}{(n+1)^2 b \delta} [w_s - w_b] > 0.
$$

(100)

$$
q_{b}^o - q_{s}^o = \frac{1}{(n+1) b} [w_s - w_b] > 0.
$$

(101)
Also,

\[
x^2_1 - x^2_2 = \frac{\delta n^2}{\delta - (n^2 + 1) \frac{r}{2}} - w_s - w_b \left[ 1 + \frac{\delta n^2}{\delta - (n^2 + 1) \frac{r}{2}} \right].
\]  

(102)

This implies that

\[
x^2_2 \geq x^2_1 \iff w_s - w_b \leq \frac{n^2}{(n^2 + 1) \left( \delta - \frac{r}{2} \right)} (a - w_b).
\]  

(103)

Comparing: \(0 < n_s < n\) and \(n_s = n\)

From (10), (11), (27), (28), with \(i = s\), we get:

\[
x_{1s} - x^o_{1s} = \frac{[2\delta - (n^2 + 1) r]}{(n + 1)^2 b \delta (2\delta - r)} \frac{[(2 + n) \delta - r]}{(\delta - r)} \frac{n_b}{n} (w_s - w_b) > 0,
\]  

(104)

\[
x_{2s} - x^o_{2s} = \frac{[2\delta - (n^2 + 1) r]}{(n + 1)^2 b \delta (\delta - r)} \frac{n_b}{n} (w_s - w_b) > 0.
\]  

(105)

From (9) and (26), with \(i = s\), it comes

\[
q^c_s - q^o_s = -\frac{n}{(n + 1) b} \frac{n_b}{n} (w_s - w_b) < 0.
\]  

(106)

Comparing: \(0 < n_b < n\) and \(n_b = n\)

From (10), (11), (27), (28), with \(i = b\), we get:

\[
x_{1b} - x^o_{1b} = -\frac{[2\delta - (n^2 + 1) r]}{(n + 1)^2 b \delta (2\delta - r)} \frac{[(2 + n) \delta - r]}{(\delta - r)} \frac{n_s}{n} (w_s - w_b) < 0,
\]  

(107)
\[ x_{2b} - x_{2b}^o = - \left[ \frac{2\delta - (n^2 + 1)r}{(n + 1)^2 b\delta (\delta - r)} n \right] \frac{n_s}{n} (w_s - w_b) < 0. \]  

(108)

From (9) and (26), with \( i = b \), it comes

\[ q_b^o - q_b^o = \frac{n}{(n + 1) b} n \frac{n_s}{n} (w_s - w_b) > 0. \]  

(109)

Since \( x_{1s} \) and \( x_{2s} \) are the two important threshold stock levels for the big firm strategies in the asymmetric case, we need to compare them to the corresponding thresholds of the symmetric case: \( x_{1b}^o \) and \( x_{2b}^o \). We have:

\[ x_{1s} - x_{1b}^o = \frac{2\delta - (n^2 + 1)r}{(n + 1)^2 b\delta (2\delta - r)} (w_s - w_b) \left[ \frac{n_b}{n} \frac{(2 + n)\delta - r}{(\delta - r)} - 1 \right], \]  

(110)

from which we get:

\[ x_{1s} - x_{1b}^o \geq 0 \iff \frac{n_b}{n} \geq \frac{1}{1 + \frac{(n+1)\delta}{\delta - r}} \equiv \rho^o. \]  

(111)

We have

\[ \rho^o \equiv \frac{1}{1 + \frac{(n+1)\delta}{\delta - r}} = \frac{\delta - r}{(2 + n)\delta - r}. \]  

(112)

We also have

\[ x_{2s} - x_{2b}^o = \frac{(n^2 + 1)}{(n + 1)^2 b\delta} (w_s - w_b) \left[ \frac{n_b}{n} \frac{2\delta - (n^2 + 1)r}{n (n^2 + 1)(\delta - r)} - 1 \right]. \]  

(113)
Then,

\[ x_{2s} - x_{2b}^o \geq 0 \iff \frac{n_b}{n} \left[ \frac{2\delta - (n^2 + 1) r}{(n^2 + 1)(\delta - r)} \right] \geq 1 \tag{114} \]

or

\[ x_{2s} - x_{2b}^o \geq 0 \iff \frac{n_b}{n} \left[ 1 - \frac{(n^2 - 1) \delta}{(n^2 + 1)(\delta - r)} \right] \geq 1, \tag{115} \]

which gives

\[ x_{2s} - x_{2b}^o \geq 0 \iff \frac{n_b}{n} \geq \frac{1}{1 - \frac{(n^2 - 1) \delta}{(n^2 + 1)(\delta - r)}} \equiv \rho_2^o. \tag{116} \]

However, since \( \frac{n_b}{n} < 1 \), and that

\[ \rho_2^o \equiv \frac{(n^2 + 1)(\delta - r)}{[2\delta - (n^2 + 1) r]} = \frac{1}{1 - \frac{(n^2 - 1) \delta}{(n^2 + 1)(\delta - r)}} \geq 1, \tag{117} \]

then the first inequality in (116) cannot happen and then finally, we will always have

\[ x_{2s} - x_{2b}^o \leq 0. \tag{118} \]

### 7.4 APPENDIX D: Derivations for Section 5

**Regular or Irregular steady-state stock at** \( x^* = x_{1s} \)?

We have an irregular steady-state stock at \( x_{1s} \) if \( \Delta_1 \equiv \tilde{x} - x_{1s} \leq 0 \). Putting \( \Delta_1 = 0 \) allows to define the locus that splits the \((\rho, \varepsilon)\)-space into two parts: one in which \( x^* = x_{1s} \) i.e. we have an irregular steady state which is \( x_{1s} \) and one in which \( x^* = \tilde{x} \), i.e. a regular steady state (see Figures
Note that:

\[
\frac{n\alpha}{(n\alpha - \delta)} = \frac{(n + 1) (\delta - \tfrac{r}{2})}{\delta - (n + 1) \tfrac{r}{2}} = 1 + \frac{n\delta}{\delta - (n + 1) \frac{r}{2}}.
\]

We can simplify (42), using (36) to get:

\[
\tilde{x} = \frac{[\delta - (n^2 + 1) \frac{r}{2}]}{[\delta - (n + 1) \frac{r}{2}] (n + 1) b\delta} \left[ a - \bar{w} + \left( \rho - \frac{1}{2} \right) \varepsilon \right].
\]

(119)

Equalizing this last expression with (34), we get:

\[
\frac{[a - \bar{w} + (\rho - \frac{1}{2}) \varepsilon]}{[\delta - (n + 1) \frac{r}{2}]} = \left\{ a - \bar{w} + \left[ \frac{(2 + n)\delta - r}{(\delta - r)} \rho - \frac{1}{2} \right] \varepsilon \right\}
\]

from which we deduce

\[
\Delta_1 \equiv \tilde{x} - x_{1s} \leq 0 \iff \left\{ \frac{1}{2} + \left[ \frac{(n + 1) (\delta - (n + 1) \frac{r}{2})}{\delta - r} - 1 \right] \rho \right\} \varepsilon \geq (a - \bar{w})
\]

or

\[
\Delta_1 \equiv \tilde{x} - x_{1s} \leq 0 \iff \left\{ \frac{1}{2} + \left[ \frac{\delta - (n^2 + 1) \frac{r}{2}}{n (\delta - r)} \right] \rho \right\} \varepsilon \geq (a - \bar{w}).
\]

Finally, we get

\[
\Delta_1 \equiv \tilde{x} - x_{1s} \leq 0 \iff \varepsilon \geq \frac{(a - \bar{w})}{\frac{r}{2} + \tau_1 \rho} \equiv E_1 (\rho),
\]

(120)
where

\[ \tau_1 \equiv \frac{1}{n} \left( \frac{\delta - (n^2 + 1) \frac{r}{\delta - r}}{\delta - (n + 1) \frac{r}{\delta - r}} \right) > 0. \quad (121) \]

Equation (120) shows that in the \((\rho, \varepsilon)\)-space, above and all along the locus depicted by the equation \(\varepsilon = E_1(\rho)\), we have \(\Delta_1 \leq 0\), that is we have an irregular steady-state stock which is \(x_{1s}\). Below that locus, we have a regular steady-state stock which is \(\tilde{x}\).

**Regular or Irregular steady-state stock at \(x^* = x_{2s}\)?**

Following the same steps as previously, we have an irregular steady-state stock at \(x_{2s}\) if \(\Delta_2 \equiv \tilde{x} - x_{2s} \geq 0\). The limiting case \(\Delta_2 = 0\) will allow us to define the locus that splits the \((\rho, \varepsilon)\)-space in two parts: one in which \(x^* = x_{2s}\) i.e. we have an irregular steady state which is \(x_{2s}\) and one in which \(x^* = \tilde{x}\), i.e. a regular steady state (see Figures 12 and 13).

Then equalizing (119) and (40), we get:

\[
\left[ \frac{\delta - (n^2 + 1) \frac{r}{\delta - r}}{\delta - (n + 1) \frac{r}{\delta - r}} \right] \left[ a - \bar{w} + \left( \frac{\rho - \frac{1}{2}}{\delta - (n + 1) \frac{r}{\delta - r}} \right) \varepsilon \right] = \frac{(n^2 + 1)}{(n + 1)} \left\{ a - \bar{w} + \left[ \frac{2\delta - (n^2 + 1) r}{(n^2 + 1) (\delta - r)} \right] \left( \frac{1}{\rho} - \frac{1}{2} \right) \varepsilon \right\},
\]

from which we deduce that \(\Delta_2 \equiv \tilde{x} - x_{2s} \geq 0\) if and only if:

\[
\left\{ \frac{1}{2} n (n - 1) \delta + \left[ -n (n - 1) \delta + (n^2 - 1) \left( \delta - (n + 1) \frac{r}{\delta - r} \right) \frac{\rho}{\delta - r} \right] \right\} \varepsilon \\
\geq \left\{ n (n - 1) \delta \right\} (a - \bar{w}),
\]

and after further simplifications,

\[
\Delta_2 \equiv \tilde{x} - x_{2s} \geq 0 \iff \left\{ \frac{1}{2} + \left[ \frac{\delta - (n + 1) \frac{r}{\delta - r}}{n} \right] \right\} \varepsilon \geq (a - \bar{w}),
\]
\[ \Delta_2 \equiv \tilde{x} - x_{2s} \geq 0 \iff \left\{ \frac{1}{2} + \left[ \frac{\delta - (n^2 + 1) \bar{r}}{n(\delta - r)} \right] \rho \right\} \varepsilon \geq (a - \bar{w}), \]

and finally

\[ \Delta_2 \equiv \tilde{x} - x_{2s} \geq 0 \iff \varepsilon \geq \frac{(a - \bar{w})}{\frac{1}{2} + \tau_2 \rho} \equiv E_2(\rho), \quad (122) \]

where

\[ \tau_2 \equiv \frac{1}{n} \frac{\delta - (n^2 + 1) \bar{r}}{n - (\delta - r)} > 0. \quad (123) \]

Equation (122) shows that in the \((\rho, \varepsilon)\)-space, above and all along the locus depicted by the equation \( \varepsilon = E_2(\rho) \), we have \( \Delta_2 \geq 0 \), that is we have an irregular steady-state stock which is \( x_{2s} \). Below that locus, we have a regular steady-state stock which is \( \tilde{x} \).

**Remark:**

We note that, \( \tau_2 = \tau_1 \). This implies that \( E_2(\rho) = E_1(\rho) \). In other terms, the locus depicted by equations \( \varepsilon = E_1(\rho) \) and \( \varepsilon = E_2(\rho) \) coincide. Above this common locus we have an irregular steady-state stock, either \( x_{1s} \) or \( x_{2s} \) and below, we have a regular steady-state stock \( \tilde{x} \). However, above that locus, it is not possible to make the distinction between the irregular steady-state stocks \( x_{1s} \) and \( x_{2s} \). We will then refer to that locus as being depicted by equation \( \varepsilon = E(\rho) \), where \( E(\rho) = E_1(\rho) = E_2(\rho) \) and then the corresponding \( \tau \) is given by \( \tau = \tau_1 = \tau_2 \).

**Two or No steady-state stock Over Interval** \([x_{2s}, x_{\text{max}}]\)?

We have two (regular) steady-state stocks over \([x_{2s}, x_{\text{max}}]\) if \( \Delta_3 \equiv g(x_{\text{max}}/2) - Q^c \geq 0 \). The limiting case \( \Delta_3 = 0 \) allows us to define the locus that splits the \((\rho, \varepsilon)\)-space in two parts: one in
which there are two steady states over \([x_{2s}, x_{\text{max}}]\) and one in which there is no steady state at all.\(^{10}\) (see Figures 12 and 13).

Using (39) and (26), we can rewrite \(Q^c\) as

\[
Q^c = Q_{1/2}^{\text{co}} + \frac{n}{(n+1)b} \left( \rho - \frac{1}{2} \right) \varepsilon,
\]

where

\[
Q_{1/2}^{\text{co}} = \frac{n(a - \bar{w})}{(n+1)b}.
\]

\(Q_{1/2}^{\text{co}}\) is the static Cournot aggregate production when both types firms are either in equal number (i.e. \(\rho = \frac{1}{2}\)) or when all firms are identical as in Benchekroun’s model, with a marginal cost equal to \(\bar{w}\). Then, we can rewrite:

\[
\Delta_3 = g \left( \frac{x_{\text{max}}}{2} \right) - Q_{1/2}^{\text{co}} - \frac{n}{(n+1)b} \left( \rho - \frac{1}{2} \right) \varepsilon.
\]

Consequently,

\[
\Delta_3 \geq 0 \iff \varepsilon \leq \frac{(n+1)b}{n} \left[ g \left( \frac{x_{\text{max}}}{2} \right) - Q_{1/2}^{\text{co}} \right] \frac{1}{\left( \rho - \frac{1}{2} \right)} \equiv E_3 (\rho).
\]

The shape of the locus depicted by equation \(\varepsilon = E_3 (\rho)\) depends on the sign of \(g \left( \frac{x_{\text{max}}}{2} \right) - Q_{1/2}^{\text{co}}\).

Below that locus, there are two (regular) steady-state stocks over \([x_{2s}, x_{\text{max}}]\) and above, no steady-state stock. Along that locus, both steady-state stocks coincide.

\(^{10}\)Note that along that locus, we have a particular case in which both steady states stock coincide in one.
Note that,

\[
\left[ g\left( \frac{x_{\text{max}}}{2} \right) - Q^{co}_{1/2} \right] \geq 0 \iff \delta \geq \frac{2n}{(n + 1)b} \frac{(a - \bar{w})}{x_{\text{max}}} = \delta^o.
\]
8 Figures
Figure 1: Equilibrium Strategies and Aggregate Outcome
when $0 < n_s, n_b < n$
Figure 2: Equilibrium Strategies and Aggregate Outcomes
when $n_b = n$ or $n_s = n$
Figure 3: Equilibrium Strategies and Aggregate Outcomes
when $0 < n_s < n$ or $n_s = n$
Figure 4: Equilibrium Strategies and Aggregate Outcomes
when $0 < n_b < n$ or $n_b = n$
Figure 5: Equilibrium Strategies and Aggregate Outcomes
when $0 < n_x, n_b < n$, $n_b = n$, or $n_x = n$
Figure 6: Steady States when $x_{1s} < \tilde{x} < x_{2s}$ and $g(x_{\text{max}}/2) \geq Q^c$

Three Steady States (All Regular)
Figure 7: Steady States when $x_{ls} < \bar{x} < x_{2s}$ and $g(\frac{x_{max}}{2}) < Q^c$  
One Steady State (Regular)
Figure 8: Steady States when $\bar{x} \geq x_{s2}$ and $g\left(x_{\text{max}}/2\right) \geq Q^c$

Three Steady States (One Irregular)
Figure 9: Steady States when $\bar{x} \geq x_{2y}$ and $g\left(\frac{x_{\text{max}}}{2}\right) < Q^c$

One Steady State (Irregular)
Figure 10: Steady States when \( \tilde{x} \leq x_{1s} \) and \( g\left(\frac{x_{\text{max}}}{2}\right) \geq Q^c \)

Three Steady States (One Irregular)
Figure 11: Steady States when $\tilde{x} \leq x_{1s}$ and $g(x_{\text{max}}/2) < Q^c$

One Steady State (Irregular)
Figure 12: Steady States in the $(\rho, \varepsilon)$-Space when $\delta > \delta^o$

Figure 13: Steady States in the $(\rho, \varepsilon)$-Space when $\delta < \delta^o$
\[ E = E(\rho) \]
\[ \Delta_1 = 0, \Delta_2 = 0 \]

One Steady State (Irregular)

One Steady State (Regular)

Three Steady States (of which one irregular)

Three Steady States (all regular)

Figure 13: Steady States in the \((\rho, \varepsilon)\)-Space when \(\delta < \delta^o\)
References


