Recapitalization vs. Liquidation
in Resolution of Financial Distress

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Abstract

This paper aims to provide a formal analysis on the choice between liquidation and recapitalization when firms get into financial difficulty. Two questions are studied here: under which conditions are the firms recapitalized or liquidated? Is the recapitalization option superior to the liquidation option? These questions are addressed by introducing the recapitalization possibility into a model of dynamic financial contracting under moral hazard. That means, instead of assuming that the entrepreneur is liable for payments to the investors only to the extent of current revenues like previous papers, in this paper, we allow the lower bound for the entrepreneur’s compensation to be a strictly negative number. We find that the firm is never recapitalized nor liquidated in a period in which the high cash flow is realized. These two options are resorted to after a poor performance when the entrepreneur’s share to the future cash flows is low enough. Hence, in our paper, the financial distress region corresponds to the situation where the entrepreneur is provided with a low stake to the future cash flows and so, weak incentives to manage the project. Another conclusion is that if the project’s profitability is high, the recapitalization option is preferred to the liquidation one in the punishment system. (JEL classifications: D82, G33, G35)

1 Introduction

In general, two options are available to financially distressed firms. The first one corresponds to the liquidation. In strict sense of the term, the liquidation refers to the process by which the existence of a firm is brought to an end. Its assets are disposed of piece by piece in order to repay the claimants in accordance with their priority rights. Alternatively, the firm may embark on a reorganization whose purpose is to find a method to overcome the trouble. Typically, it involves a process of negotiation between debtors and creditors with a view to establishing a new mechanism for the settlement of claims: writting off some of the claims, exchanging bonds and other debts with new notes, bonds, swapping

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new equities for old ones, injection of new capital (recapitalisation). This paper aims at providing a formal analysis on the choice between liquidation and recapitalization - one of possible solutions involved in reorganization process - when firms get into financial difficulty. Two questions are studied here: under which conditions are the firms recapitalized or liquidated? Is the recapitalization option superior to the liquidation option?

We take a dynamic financial contracting approach to these questions. The dynamic agency model is first introduced by Green (1987) and Spear and Srivastava (1987). Built on the recursive method developed in these works, recently, a number of papers (e.g. DeMarzo and Fishman (2004; 2006), Biais et al. (2004; 2006), Clementi and Hopenhayn (2006)) study optimal financial contract in a setting in which a risk neutral agent seeks funding from risk neutral investors (principal) to finance a project that pays stochastic cash flows over many periods. Their contracting relationship is subject to a moral hazard problem coming either from the unobservability of cash flows or from the hidden effort. All of these papers assume that the agent is liable for payments to the investors only to the extent of current revenues. As a consequence, the agent’s compensation at each period must be non negative. The investors thus possess in hand two incentive devices: cash reward or threat of liquidation. In this paper, we also consider a dynamic moral hazard model with risk neutral agents. However, to be able to analyze simultaneously the liquidation and the recapitalization decisions, we relax the limited liability condition of previous analysis by assuming that the lower bound for the agent’s payment is a strictly negative number.

More concretely, in our model, there are two types of agents: an entrepreneur and investors. All are risk neutral and discount the future by the same rate. The entrepreneur is endowed with a multi-period project that requires an initial investment of I. The project, once funded, generates at each date t=1, 2 ...T an observable binary cash flows whose distribution depends on the effort of the entrepreneur. The choice of effort is unobservable to the investors. For simplicity, we assume that the set of feasible effort levels contains two elements, i.e. the entrepreneur can either exert high effort or low effort. The distribution of cash flows under high effort dominates the one under low effort in the sense of first order stochastic dominance. Nevertheless, exerting high effort is costly since it deprives the entrepreneur of a private benefit B. We assume that the current effort only affects the current cash flows. This means that the cash flows are identically and independently distributed over time.

The entrepreneur seeks to finance her project by contracting with the investors. A long term financing contract between them specifies at each date t:

i) First, the probability of continuation determined by the investors. The liquidation value of the project is assumed to be equal to 0.

ii) Second, the entrepreneur’s compensation conditional on the project not being liquidated. Unlike the previous literature, in this paper, we allow for negative values of the entrepreneur’s compensation, i.e. it is bounded below by an exogenous negative number –K. Hence, in our model, at each date t, the investors can require the entrepreneur to inject some more capital. Negative compensation will be interpreted as recapitalization. To make our extension non trivial, we introduce a cost of recapitalization. In fact, the
recapitalization costs contain some elements that are fixed and others which are proportional. The fixed elements can be explained by transaction fees. The proportional costs may result from dilution effects as in Mayer - Majluf (1984). In this paper, for simplicity, we only take into account the proportional costs and assume that it is exogenous instead of modeling it.

Some remarks deserve to be noted concerning our modeling of the recapitalization decision. First, in DeMarzo and Fishman (2006) as well as in Clementi and Hopenhayn (2006), at each period, some amount of money can be injected by the investors in the project. This capital injection results in an expansion of the firm’s size and is interpreted as investment. In our model, the capital injection is realized by the entrepreneur. It does not imply a change of the firm’s size but it serves as an additional source of fund beside the cash flows to repay debt so that the firm can be continued. These properties are consistent with the purpose of the recapitalization in the period of financial distress. Second, since in our model, the recapitalization decision is determined as a part of the optimal design of the lending contract, the recapitalization threshold we obtain is endogenous. It differs from several papers which set this threshold in exogenous way.

In line with numerous analysis of the literature on repeated moral hazard, to characterize the optimal contract, we use the expected discounted utility of the entrepreneur at the start of each period as the only state variable. We begin by considering the case where the project lives in two periods (i.e. T=2), the liquidation value is set exogenously at zero and contracting parties can commit not to renegotiate the contract. Under this simple setting, we are able to fully solve for the optimal contract and explicitly characterize different thresholds corresponding to different policies. According to the optimal contract, dividend is paid only when the expected discounted value of the cash flows accruing to the entrepreneur reaches a high level. Following a good performance of the firm, neither recapitalization nor liquidation are observed. After a low realization of the cash flows, recapitalization and liquidation are resorted to when the entrepreneur’s stake to the future cash flows falls below a threshold. Therefore, the financial distress corresponds to the situation in which the entrepreneur is provided with a low stake to the future cash flows and so, weak incentives to manage the project. The threshold for the financial distress region is increasing in the magnitude of moral hazard problem and decreasing in the recapitalization cost. Relatively to the question whether the recapitalization option is superior to the liquidation one, our result shows that there is no strict priority order between these two instruments. When the profitability of the project is high, recapitalization is preferred to liquidation in the punishment system. In other case, this order is reversed.

Next, to confirm the robustness of our results, we extend the basic setting in different directions. We allow for the possibility of renegotiating the contract and the infinite horizon of the project’s life.

2 Related literature

This paper typically links to the literature on dynamic financial contracting under moral hazard. The most recent contributions consist of DeMarzo and Fishman (2004;
Biais et al. (2004; 2006), Clementi and Hopenhayn (2006). They examine how financial contracts can be designed to solving the asymmetric information problem between contracting parties which repeatedly interact. The moral hazard problems considered in these papers may be an unobservable cash-flow (ex-post moral hazard) or an unobservable effort (ex-ante moral hazard). For example, DeMarzo and Fishman (2004) look at the optimal long-term financial contract in a setting in which a risk-neutral agent with limited capital seeks funding from risk-neutral investors to finance a project that pays stochastic cash flows over many periods. These cash flows are observable to the agent but not to investors. So, the investors’ concern is to design a contract that may deter the agent from diverting funds to himself. DeMarzo and Fishman (2004) fully characterize the optimal contract and show that the optimal mechanism can be implemented by a combination of equity, long-term debt and a line of credit - commonly observed securities. The long-term debt in their implementation is described by a fixed charge paid at each period. The credit line is characterized by an interest rate and a credit limit. The dividend is paid to equityholders if the business cash-flows exceed the required debt and credit line payments.

As for Biais et al. (2004), they analyze another case of moral hazard: unobservable effort. Their model can be seen as a generalization of static model of Holmstrom and Tirole (1997). In their model, a risk neutral entrepreneur contacts risk neutral financiers to fund an investment project that yields random observable and binary cash-flows at each period. The distribution of cash-flows depends on the effort of the entrepreneur which is not observable to investors. Concretely, the distribution of cash-flows under high effort first order stochastically dominates the distribution of cash-flows under low effort. Exerting high effort is costly to the entrepreneur because she will lose a private benefit B. In this setting, Biais et al. (2004) analyze the optimal contract specifying the payment to the entrepreneur and the financiers as well as the decision to liquidate the project for two cases: identical and different discount rates of entrepreneur and financiers. They also consider the implementation of the optimal contract but their implementation is realized via debt, equity and cash reserves. Their endogenous securities are defined as streams of cash-flows. Debt is a claim on a steady stream of constant coupons. Equity is a claim on a more irregular stream of cash, paid only after accumulated earnings have reached a threshold. Another contribution of their paper lies in considering the continuous time limit of discrete time model to derive a rich set of asset pricing implications.

Another paper related to ours is Shim (2004) which is an application of dynamic contracting literature to banking regulation. In today’s world, financial innovation has produced new markets and instruments that make it easy for banks and their employees to make huge bets easily and quickly. This change in the financial environment for banking institutions has lead regulators and economists to pay more attention to the importance of supervisory review in achieving the safety and soundness of banking system. In order to implement the regulatory supervision in practice, academic research focuses on the threshold and form of intervention. Basel II remains silent on this issue. The US FDICIA of 1991 introduces early and gradual intervention rules, called PCA. It specifies five capital/asset zones and imposition of regulatory restraints on bank is capital decrease below an predetermined established number. If a bank is classified as well or adequately capital-
ized, it is not subject to any form of intervention. On the other hand, undercapitalized banks are subject to increasingly severe mandatory sanctions as their capital deteriorates. For all undercapitalized banks, constraints are imposed such as restrictions on distributing dividends and expanding total assets. Significantly undercapitalized banks must restore capital by selling stocks or be merged. Critically undercapitalized banks are subject to liquidation. Shim (2004) analyzed the welfare properties of this PCA. His methodology lies in adapting the DeMarzo-Fishman’s (2004) dynamic model of entrepreneurial finance to banking regulation. Specifically, he first constructs a dynamic model and characterizes the optimal allocation between a risk-neutral banker and a risk-neutral regulator (FDIC) under hidden choice of effort, private information on returns and limited commitment by the banker. Then, he demonstrates that this allocation can be implemented through a combination of two regulatory tools: book-value capital regulation and risk-based deposit insurance. Hence, in Shim (2004), banking regulation is seen as an instrument to implement the optimal contract between the banker and the FDIC. His conclusion is that it is optimal to base bank capital regulation on book-value capital as in PCA but the optimal allocation suggests that stochastic termination is better than deterministic liquidation.

3 Model

3.1 Environment

This model is a dynamic effort-problem model adapted from Biais et al. (2004). There are two types of agents: an entrepreneur and investors. All are risk neutral and discount the future by the same rate \( r^1 \).

The entrepreneur is endowed with a multi-period project that requires an initial investment of \( I \). The project, if financed, generates at each date \( t = 1, 2, ..., T \), an observable cash flow \( R_t \) that can take two possible values \( R_t = R_+ \) in case of success and \( R_t = R_- \) in case of failure. The distribution of cash flow depends on the effort \( e_t \) of the entrepreneur to monitor her project. The choice of effort is unobservable to the investors. For simplicity, we assume that the entrepreneur can either exert high effort \( (e_t = 1) \) or low effort \( (e_t = 0) \). If she chooses high effort, the probability of success is \( p > p^- \Delta p \) which is the probability of success in case of low effort. However, if the entrepreneur decides not to exert high effort, she gets a private benefit \( B \). Cash flows are assumed to be independently distributed across periods.

To raise the lacking capital, the entrepreneur will have to enter into a contractual relationship with investors. A long term financing contract between them specifies at each date \( t \):

i) first, the probability of continuation determined by the investors \( x_t \). The liquidation value of the project is assumed to be equal to 0.

ii) second, the entrepreneur’s consumption \( c_t \) conditional on the project not being liquidated. Differently from Biais et al. (2004), we allow for negative values of \( c_t \): \( c_t \geq -K \) where \( K > 0 \). Hence, in our model, at each date \( t \) investors can require the

\(^1\)Biais et al. (2004) consider also the case of different discount rates
entrepreneur to inject some more cash. Negative $c_t$ can be interpreted as recapitalisation. To make our extension non trivial, we introduce a cost of recapitalisation $\tau > 0$. The utility function of the entrepreneur thus takes the following form:

$$U(c_t) = \begin{cases} c_t & \text{if } c_t \geq 0 \\ (1 + \tau) c_t & \text{if } c_t \in [-K, 0) \end{cases} \quad (1)$$

Therefore, in our paper, the cost of recapitalisation is proportionally introduced. In fact, the recapitalisation costs contain some elements that are fixed and others which are proportional. The fixed elements can be explained by transactions fees. The proportional costs may result from dilution effects as in Mayer - Majluf (1984). In this paper, for simplicity, we only take into account the proportional costs and assume that $K$ and $\tau$ are exogenously given numbers. $K$ and $\tau$ can be considered as a measure of the magnitude of the capital market imperfections. We denote:

$$\bar{R} = pR_+ + (1 - p)R_-$$
$$R = (p - \Delta p)R_+ + (1 - p + \Delta p)R_-$$

**Assumption 1**

$$\bar{R} > 0 > R + B$$

i.e. the project is profitable only if the entrepreneur carefully monitors it.

**Assumption 2**

$$\frac{pB}{\Delta p} > (1 + \tau)K$$

this assumption ensures that the agency problem is non trivial in this setting.

**Assumption 3**

$$\bar{R} \geq (1 - p) \frac{\tau pB}{1 + \tau \Delta p}$$

### 3.2 Dynamic programming problem

As is standard in repeated moral hazard model, the optimal contract is best characterized through a state variable $w_t$ representing the expected discounted utility of the entrepreneur at the beginning of date $t$. Hence, at each date $t$, the investors must choose, as a function of $w_t$, a probability of continuation $x_t$, the consumptions to the entrepreneur $\xi_t = c_t(R_+)$ and $\zeta_t = c_t(R_-)$ and her contingent continuation utilities $\bar{w}_{t+1}$ and $\bar{w}_{t+1}$. We will focus on the contract that induces the entrepreneur to exert high effort in all periods. The choice of the variables $(\bar{\xi}_t, \bar{\zeta}_t, \bar{w}_{t+1}, \bar{w}_{t+1}, x_t)$ by the investors must satisfy the following constraints:

First, the promise keeping constraint states that what the entrepreneur was expecting to receive at the beginning of date $t$ must be equal to the sum of the utility she derives from the consumption paid to her during this period and the expected present value of
her continuation utility:

\[ x_t \left[ pU(t) + (1 - p)U({\xi}) + \frac{p\bar{w}_{t+1} + (1 - p)w_{t+1}}{1 + r} \right] = w_t \]  

(2)

Second, the entrepreneur must be given incentive to exert high effort. To do that, the expected utility the entrepreneur gets under high effort must be greater than her expected utility under low effort:

\[ x_t \left[ pU(t) + (1 - p)U({\xi}) + \frac{p\bar{w}_{t+1} + (1 - p)w_{t+1}}{1 + r} \right] \geq x_t \left[ (p - \Delta p)U(t) + (1 - p + \Delta p)U({\xi}) + \frac{(p - \Delta p)\bar{w}_{t+1} + (1 - p + \Delta p)w_{t+1} + B}{1 + r} \right] \]

i.e.

\[ U(t) - U({\xi}) + \frac{\bar{w}_{t+1} - w_{t+1}}{1 + r} \geq \frac{B}{\Delta p} \]  

(3)

Hence, managerial incentives can be provided either through the promise of continuation or the threat of termination or through financial compensation.

Third, these variables have to satisfy some feasibility conditions:

\[ x_t \in [0; 1] \]  

(4)

\[ (t, \xi) \in [-K, +\infty)^2 \]  

(5)

\[ (\bar{w}_{t+1}, w_{t+1}) \in R^2_+ \]  

(6)

Conditions (4) and (5) come directly from the model. Condition (6) states that the entrepreneur’s continuation payoff must be at least equal to her reservation value normalized at zero. It is so called renegation-proofness constraint since it avoid any breach of contract from the entrepreneur.

Denote by \( F_t(w_t) \) the highest possible continuation utility attainable by the investors given a continuation utility \( w_t \) for the entrepreneur at the beginning of date \( t \). So, the function \( F_t \) satisfies the Bellman equation:

\[ F_t(w_t) = \max \left[ \begin{array}{l}
\bar{R} - p\bar{c}_t - (1 - p)\xi_t + \frac{pF_{t+1}(\bar{w}_{t+1}) + (1 - p)F_{t+1}(w_{t+1})}{1 + r} \end{array} \right] \]  

(7)

for all \( w_t \geq 0 \), subject to constraints (2) – (6). Following Biais et al. (2004), we introduce an auxiliary function defined by \( V_t(w_t) = F_t(w_t) + w_t \). \( V_t(w_t) \) represents the expected social surplus generated by the project from date \( t \) on and it will satisfy the Bellman:

\[ V_t(w_t) = \max \left[ \begin{array}{l}
\bar{R} + p(U(t) - \bar{c}_t) + (1 - p)(U({\xi}) - \xi_t) + \frac{pV_{t+1}(\bar{w}_{t+1}) + (1 - p)V_{t+1}(w_{t+1})}{1 + r} \end{array} \right] \]  

for all \( w_t \geq 0 \) subject to constraints (2) – (6).

4 Two periods setting
In this section, we consider the case where the project lives in two dates. The timeline is shown in the following figure:

If the investors agree to finance the initial investment $I$, a financing contract is concluded between them and the entrepreneur. This contract will specify financial compensations paid to the entrepreneur $c_{1}, c_{2}$ and the continuation probability $x_{2}$. To solve for the optimal contract, we use backward induction. We begin by finding the continuation contract for the second period. This contract depends on the expected utility $w_{2}$ promised to the entrepreneur at the start of second period. Since period 2 is the final period, continuation is no more an option. Therefore, the promise keeping constraint and the incentive compatibility constraint become

$$x_{2} [pU(\tau_{2}) + (1-p)U(\xi_{2})] = w_{2} \quad \text{(8)}$$

$$U(\tau_{2}) - U(\xi_{2}) \geq \frac{B}{\Delta p} \quad \text{(9)}$$

The continuation contract is then determined by the program

$$V_{2}(w_{2}) = \max_{x_{2},\tau_{2},\xi_{2}} x_{2} \left[ \bar{R} + p (U(\tau_{2}) - \tau_{2}) + (1-p) (U(\xi_{2}) - \xi_{2}) \right]$$

for all $w_{2} \geq 0$ s.t (8) – (9) and feasibility condition $(x_{2}, \tau_{2}, \xi_{2}) \in [0,1] \times [-K, +\infty)^2$.

$V_{2}(w_{2})$ measures the continuation value of the project. Its value and the optimal choices for $(x_{2}, \tau_{2}, \xi_{2})$ are characterized in the lemma 1$^2$.

**Lemma 1** the continuation contract is characterized as follows:

- For $w_{2} \geq \frac{pB}{\Delta p}$, the project is continued with certainty $(x_{2} = 1)$. After the second-period cashflow is realized, the entrepreneur receives $\overline{c}_{2} = w_{2} + \frac{(1-p)B}{\Delta p} > 0$ in case of success and $\xi_{2} = w_{2} - \frac{B}{\Delta p} \geq 0$ in case of failure. $V_{2}(w_{2}) = \bar{R}$.

- For $\frac{pB}{\Delta p} - (1+\tau)K \leq w_{2} < \frac{B}{\Delta p}$, the project is also continued with certainty $(x_{2} = 1)$. In the second period, the entrepreneur gets a positive compensation $\overline{c}_{2} = w_{2} + \frac{(1-p)B}{\Delta p}$

$^2$See appendix 1 for the proof.
in case of success and needs to recapitalise in case of failure if
\[ c_2 = \frac{1}{1 + r} \left( w_2 - \frac{pB}{\Delta p} \right) < 0. \]

\[ V_2(w_2) = (1 - p) \frac{r}{1 + r} w_2 + \tilde{R} - (1 - p) \frac{r}{1 + r} \frac{pB}{\Delta p}. \]

- For \( 0 \leq w_2 < \frac{pB}{\Delta p} - (1 + \tau)K \), the project is continued with probability \( x_2 = \frac{w_2}{\frac{pB}{\Delta p} - (1 + \tau)K} < 1 \). If the project is operated, a positive transfer paid to the entrepreneur in case of success \( \bar{c}_2 = \frac{B}{\Delta p} - (1 + \tau)K \), but in case of failure the entrepreneur must to inject maximal cashes \( c_2 = -K \). \( V_2(w_2) = \frac{\tilde{R} - (1 - p)\tau K}{\frac{pB}{\Delta p} - (1 + \tau)K} w_2 \).

This lemma reflects the idea that liquidation is a necessary punishing device if the contract must make the entrepreneur sufficiently poor in period 2 (her expected utility is sufficiently low).

The optimal continuation of the contract for the second period being defined, we can move backward to the first period. Let \( w_1 \geq 0 \) be the expected utility for the entrepreneur if the project is operated. First, we analyse the optimal contract as a function of \( w_1 \). Then, we will explain how to determine the optimal value of \( w_1 \).

Given \( w_1 \), the social surplus generated by the project is defined by

\[ V_1(w_1) = \max_{\bar{c}_1, \bar{w}_1, \bar{w}_2} \bar{R} + p[U(\bar{c}_1) - \bar{c}_1] + (1 - p)[U(\bar{c}_1) - \bar{c}_1] + \frac{pV_2(w_2) + (1 - p)V_2(w_2)}{1 + r} \]

for all \( w_1 \geq 0 \) s.t

\[ pU(\bar{c}_1) + (1 - p)U(\bar{c}_1) + \frac{p\bar{w}_2 + (1 - p)w_2}{1 + r} = w_1 \quad (11) \]

\[ U(\bar{c}_1) - U(\bar{c}_1) + \frac{\bar{w}_2 - w_2}{1 + r} \geq \frac{B}{\Delta p} \]

\[ (\bar{c}_1, \bar{c}_1) \in [\bar{K}, +\infty)^2 \]

\[ (\bar{w}_2, w_2) \in R_+^2 \quad (13) \]

Thus, if the project is undertaken, it generates an expected social surplus measured by the sum of three terms: the first term \( \bar{R} \) corresponds to the current period’s expected cash flow, the second \( p[U(\bar{c}_1) - \bar{c}_1] + (1 - p)[U(\bar{c}_1) - \bar{c}_1] \) is the expected loss in social surplus and the third \( \frac{pV_2(\bar{w}_2) + (1 - p)V_2(w_2)}{1 + r} \) is the expected discounted continuation value of the project. From the constraints \((11) - (14)\), we obtain \( w_1 \geq \frac{pB}{\Delta p} - (1 + \tau)K > 0 \), this reflects the well known result that in the presence of agency problem, the entrepreneur gets a strictly positive rent.

Owing to the lemma 1, we see that the project’s expected discounted continuation value is maximal when the entrepreneur is promised a continuation utility at least equal to \( \frac{pB}{\Delta p} \) in both states of nature. Moreover, since \( \arg \max_{\bar{c}_1, \bar{c}_1} [U(\bar{c}_1) - \bar{c}_1] + (1 - p)[U(\bar{c}_1) - \bar{c}_1] = 0, +\infty)^2 \), if one can find a \((\bar{c}_1, \bar{c}_1, \bar{w}_2, w_2) \in [0, +\infty)^2 \times \left[ \frac{pB}{\Delta p}, +\infty \right)^2 \) satisfying two constraints \((11) - (12)\), then it constitutes a solution. It is immediately to check that the investors can structure the entrepreneur payoff such that \((\bar{c}_1, \bar{c}_1, \bar{w}_2, w_2) \in [0, +\infty)^2 \times \left[ \frac{pB}{\Delta p}, +\infty \right)^3 \) if and only if \( w_1 \) belongs to the interval \( \left[ \frac{pB}{\Delta p} \left( 1 + \frac{1}{1 + \tau} \right), +\infty \right) \). So, for a

\[ (\bar{c}_1, \bar{c}_1, \bar{w}_2, w_2) = \left( w_1 - \frac{pB}{\Delta p} \left( 1 + \frac{1}{1 + \tau} \right) + \frac{pB}{\Delta p}, w_1 - \frac{pB}{\Delta p} \left( 1 + \frac{1}{1 + \tau} \right) \right) \]
sufficiently high value of \( w_1 \), the optimal contract provides a positive financial reward to the entrepreneur and permit her to continue the project with certainty. The next lemma establishes first properties of the solution over the interval \( \left[ \frac{pB}{\Delta p} - (1 + \tau)K, \frac{pB}{\Delta p}(1 + \frac{1}{1+r}) \right) \).

**Lemma 2** For \( w_1 \in \left[ \frac{pB}{\Delta p} - (1 + \tau)K, \frac{pB}{\Delta p}(1 + \frac{1}{1+r}) \right) \), at the optimum, both \( (\bar{c}_1, \underline{c}_1) \) are non positive and the incentive compatibility constraint (12) is binding.

This lemma allows us to compute \( \bar{c}_1 \) and \( \underline{c}_1 \) as a function of \( \bar{w}_2, \underline{w}_2 \).

\[
\bar{c}_1 = \frac{1}{1 + \tau} \left( w_1 + \frac{(1 - p)B}{\Delta p} - \frac{\bar{w}_2}{1 + r} \right)
\]

\[
\underline{c}_1 = \frac{1}{1 + \tau} \left( w_1 - \frac{pB}{\Delta p} - \underline{w}_2 \right)
\]

Combining the constraint (14) with condition \( (\bar{c}_1, \underline{c}_1) \in [-K, 0]^2 \), we obtain two new constraints for \( (\bar{w}_2, \underline{w}_2) \):

\[
\max \left( 0, w_1 - \frac{pB}{\Delta p} \right) \leq \frac{w_2}{1 + r} \leq w_1 - \frac{pB}{\Delta p} + (1 + \tau)K \quad (15)
\]

\[
w_1 + \frac{(1 - p)B}{\Delta p} \leq \frac{\bar{w}_2}{1 + r} \leq w_1 + (1 - p)B + (1 + \tau)K \quad (16)
\]

Thus, over the interval \( \left[ \frac{pB}{\Delta p} - (1 + \tau)K, \frac{pB}{\Delta p}(1 + \frac{1}{1+r}) \right) \), the program (10) – (14) can be rewritten as follows

\[
V_1(w_1) = \max_{\frac{\bar{w}_2}{\underline{w}_2}} \frac{\tau}{1 + \tau} w_1 + \frac{p}{1 + r} \left( V_2(\bar{w}_2) - \frac{\tau}{1 + \tau} \underline{w}_2 \right) + \frac{1 - p}{1 + r} \left( V_2(\underline{w}_2) - \frac{\tau}{1 + \tau} \bar{w}_2 \right) + \bar{R}
\]

subject to constraints (15) – (16).

Note that \( V_2(w_2) - \frac{\tau}{1 + \tau} w_2 \) is decreasing once \( w_2 \geq \frac{pB}{\Delta p} - (1 + \tau)K \). For \( w_2 \in \left[ 0, \frac{pB}{\Delta p} - (1 + \tau)K \right) \), it is decreasing if \( \bar{R} < \frac{\tau}{1 + \tau} \left( \frac{pB}{\Delta p} - p(1 + \tau)K \right) \) and increasing otherwise. Since \( w_1 \geq \frac{pB}{\Delta p} - (1 + \tau)K \), we have \((1 + r) \left( w_1 + \frac{(1 - p)B}{\Delta p} \right) \) is strictly greater than \( \frac{pB}{\Delta p} - (1 + \tau)K \). Constraint (16) implies that at the optimum, \( \bar{w}_2 \) is equal to \( (1 + r) \left( w_1 + \frac{(1 - p)B}{\Delta p} \right) \) and thus \( \bar{c}_1 \) equal to zero. Relatively to \( \underline{w}_2 \), when \( V_2(w_2) - \frac{\tau}{1 + \tau} w_2 \) is always decreasing on its domain, at the optimum, one sets \( \underline{w}_2 \) equal to zero if \( w_1 < \frac{pB}{\Delta p} \) and to \( (1 + r) \left( w_1 - \frac{pB}{\Delta p} \right) \) otherwise. As a consequence, \( \underline{c}_1 \) is respectively equal to \( \frac{1}{1 + \tau} \left( w_1 - \frac{pB}{\Delta p} \right) \) less than zero and zero. For the situation where \( V_2(w_2) - \frac{\tau}{1 + \tau} w_2 \) is increasing over \( \left[ 0, \frac{pB}{\Delta p} - (1 + \tau)K \right) \) and decreasing over \( \left[ \frac{pB}{\Delta p} - (1 + \tau)K, +\infty \right) \), the solution involves

\[
w_2 = \begin{cases} 
(1 + r) \left( w_1 - \frac{pB}{\Delta p} \right) & \text{for } w_1^* \leq w_1 < w_1^{**} \\
\frac{pB}{\Delta p} - (1 + \tau)K & \text{for } w_1^{***} \leq w_1 < w_1^{**} \\
(1 + r) \left( w_1 - \frac{pB}{\Delta p} + (1 + \tau)K \right) & \text{for } \frac{pB}{\Delta p} - (1 + \tau)K \leq w_1 < w_1^{***}
\end{cases}
\]
where
\[
\begin{align*}
  w^*_1 &= \frac{pB}{\Delta p} \left( 1 + \frac{1}{1+r} \right) \\
  w^{**}_1 &= w^*_1 - \frac{(1+\tau)K}{1+r} \\
  w^{***}_1 &= w^{**}_1 - (1+\tau)K
\end{align*}
\]

Therefore, the optimal contract involves two regimes which are summarised in the following proposition:

**Proposition 1**

* if \( \hat{R} > \frac{\tau}{1+\tau} \left( \frac{pB}{\Delta p} - p(1+\tau)K \right) \) i.e. the function \( V_2(w) - \frac{\tau}{1+\tau} w \) is increasing over \( \left[ 0, \frac{pB}{\Delta p} - (1+\tau)K \right] \) and decreasing over \( \left[ \frac{pB}{\Delta p} - (1+\tau)K, +\infty \right) \), the optimal mechanism is characterized in the following table:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Probability of continuation } x^*_2 & x_2 \left( R_1 = R^+_1 \right) = 1 & x_2 \left( R_1 = R^-_1 \right) = \frac{1+r}{\frac{pB}{\Delta p} - (1+\tau)K} w_1 - (1+\tau) < 1 & x_2 \left( R_1 = R^+_1 \right) = 1 & x_2 \left( R_1 = R^+_1 \right) = 1 \\
\hline
\text{Financial payments in case of success} & \bar{\xi}_1 = 0 & \bar{\xi}_1 = 0 & \bar{\xi}_1 = 0 & \bar{\xi}_1 > 0 \\
\hline
\text{Financial payments in case of failure} & \xi_1 = -K & -K < \xi_1 < 0 & \xi_1 = 0 & \xi_1 = 0 \\
\hline
\end{array}
\]

* If \( \hat{R} < \frac{\tau}{1+\tau} \left( \frac{pB}{\Delta p} - p(1+\tau)K \right) \) i.e. the function \( V_2(w) - \frac{\tau}{1+\tau} w \) is always decreasing on its domain \( \hat{R}_1 \), the optimal contract is as follows:
According to the optimal contract, the entrepreneur is rewarded with a cash payment only after a good performance and when $w_1$ is sufficiently high. The threshold for dividend region is increasing in the magnitude of moral hazard problem and it does not depend of the recapitalisation cost ($\tau$) as well as borrowing constraint ($K$). The firm is never recapitalized nor liquidated in a period in which the high cash flow is realized. These two options are resorted to after a poor performance when $w_1$ is low enough. Hence, the firm gets into financial difficulty when the entrepreneur is provided with a low stake to the future cash flows ($w_1 < w_1^\ast$) and so, weak incentives to manage the project. The threshold for the financial distress region is increasing in the magnitude of moral hazard problem and decreasing in $\tau$ and $K$. There exists two regimes in solving for the financial distress situation.

The first one corresponds to the case in which the project is highly profitable. When the expected utility for the entrepreneur falls into $[w_1^\ast, w_1^\ast]$, the firm is reorganized by recapitalization and it emerges from this process for sure. If the entrepreneur’s utility is lower, she is also required to contribute capital in the reorganization process but the survival probability of the firm in reorganization is strictly less than one. The survival probability is an increasing function of the entrepreneur’s initial promised utility $w_1$. If one interprets $x_2$ as the firm’s size, this result means that the emerging firm has smaller size than its pre-reorganization one.

The second regime is applied to the firm with relatively low profitability compared to the cost of recapitalization and the magnitude of moral hazard problem. Once the entrepreneur’s utility is less than $w_1^\ast$, it is optimal to liquidate the firm with a positive probability. When it falls below $\frac{pB}{\Delta p} - (1 + r)K$, the firm is liquidated with certainty. Hence, in this case, the optimal contract does not involve the recapitalization.

### 5 Robustness
5.1 Renegotiation possibility

In the previous analysis, we assume a long-term commitment between the entrepreneur and the investors. We now consider the possibility that contracting parties cannot commit not to renegotiate the contract and verify whether our optimal contract is renegotiation-proof.

From the lemma 1, we see that whenever \( R < \frac{pB}{\Delta p} - (1 + p\tau) K \), the continuation function \( F_2(w) \) is always decreasing. So, there does not exist an outcome which yields a higher payoff for all. In other words, our optimal contract is robust to the renegotiation possibility.

For the case \( R > \frac{pB}{\Delta p} - (1 + p\tau) K \), the continuation function \( F_2(w) \) is increasing over \( \left[ 0, \frac{pB}{\Delta p} - (1 + \tau) K \right) \) and decreasing otherwise. Hence, a continuation contract that offers the entrepreneur a continuation utility less than \( \frac{pB}{\Delta p} - (1 + \tau) K \) is Pareto inferior. The entrepreneur and the investors will all agree to replace it by a new contract that provide the former with a utility equal to \( \frac{pB}{\Delta p} - (1 + \tau) K \). As a result, the firm is never liquidated. It is recapitalized when \( w_1 < w_1^* \).

5.2 Infinite horizon setting

Since cash flows are iid distributed, in an infinite horizon setting, the optimal contract is stationary. So, from now on, we will skip the time subscript. The dynamic programming we need to solve is the following:

\[
V(w) = \max_{x, \ell, \pi, w} \left[ R + p(U(\pi) - \ell) + (1 - p)(U(\ell) - \ell) + \frac{pV(\pi) + (1 - p)V(w)}{1 + r} \right]
\]

for all \( w \geq 0 \), s.t.

\[
x \left[ pU(\pi) + (1 - p)U(\ell) + \frac{p\pi + (1 - p)w}{1 + r} \right] = w
\]

\[
U(\pi) - U(\ell) + \frac{\pi - w}{1 + r} \geq \frac{B}{\Delta p}
\]

\[
(\pi, \ell, \pi, w) \in [-K, +\infty)^2 \times R_+^2
\]

\[
x \in [0, 1]
\]

First, we establish the existence and some properties of the function \( V^4 \).

**Lemma 3**

i) There exists an unique continuous and bounded solution \( V \) to the above program. \( V \) is non decreasing and concave.

ii) \( V \) is strictly increasing over \( [0, \frac{1 + r pB}{\Delta p}] \) and constant over \( \left( \frac{1 + r pB}{\Delta p}, +\infty \right) \).

Next, we need to characterize the solution. For \( w \geq \frac{1 + r pB}{\Delta p} \), one can immediately check

---

\(^4\)Note the similarity of our result with the one of Biais et al. (2004).
that \[
\begin{cases}
x = 1 \\
\tau = w - \frac{1+r}{r} \frac{pB}{\Delta p} + \frac{B}{\Delta p} \\
\zeta = w - \frac{1+r}{r} \frac{pB}{\Delta p} \\
\omega = w = \frac{1+r}{r} \frac{pB}{\Delta p}
\end{cases}
\]
is a solution. Over the interval \([0, \frac{1+r}{r} \frac{pB}{\Delta p}]\), we obtain the following similar result of the lemma 2

Lemma 4 For \(w \in [0, \frac{1+r}{r} \frac{pB}{\Delta p}]\), at the optimum:

i) After a high realization of cash flow, the entrepreneur’s payment is zero

ii) After a low realization of cash flow, the entrepreneur’s compensation is non positive.

It is strictly negative only if the incentive compatibility constraint is binding.

Therefore, over this interval, the optimisation problem becomes

\[
V(w) = \max_{x, \bar{w}, w} \frac{\tau}{1+\tau} w + x \left[ \tilde{R} + \frac{p}{1+r} \left( V(\bar{w}) - \frac{\tau}{1+\tau} \bar{w} \right) + \frac{1-p}{1+r} \left( V(w) - \frac{\tau}{1+\tau} w \right) \right]
\]
for all \(w \geq 0\) subject to constraints (17) – (20).

Due to the lemma 3, two cases can occur: either \(V(w) - \frac{\tau}{1+\tau} w\) is decreasing on its domain \(R_+\) or there exists a \(\bar{w} > 0\) such that \(V(w) - \frac{\tau}{1+\tau} w\) is increasing below \(\bar{w}\) and decreasing above it.

For the situation in which \(V(w) - \frac{\tau}{1+\tau} w\) is decreasing on its domain, one can prove that at the optimum, the incentive compatibility constraint (18) is binding. Therefore, from (17) and (18), we compute \((\tau, \zeta)\) as a function of \((x, \bar{w}, w)\). Combining this result with the feasibility conditions, we obtain three new constraints for \((x, \bar{w}, w)\)

\[
\max \left( 0, \frac{w}{x} - \frac{pB}{\Delta p} \right) \leq \frac{w}{1+r} - \frac{pB}{\Delta p} + (1+\tau)K
\]

\[
\frac{w}{x} + \frac{(1-p)B}{\Delta p} \leq \bar{w} \leq \frac{w}{x} + \frac{(1-p)B}{\Delta p} + (1+\tau)K
\]

\[
0 \leq x \leq \inf \left( 1, \frac{w}{\frac{pB}{\Delta p} - (1+\tau)K} \right)
\]

Since \(V(w) - \frac{\tau}{1+\tau} w\) is a decreasing function, at the optimum, we have

\[
\bar{w} = (1+r) \left( \frac{w}{x} + \frac{(1-p)B}{\Delta p} \right)
\]

\[
w = (1+r) \max \left( 0, \frac{w}{x} - \frac{pB}{\Delta p} \right)
\]

Because the optimal values of \((\bar{w}, w)\) is decreasing with \(x\), the objective function is thus increasing with \(x\). This means that the optimal liquidation policy involves \(x = \inf \left( 1, \frac{w}{\frac{pB}{\Delta p} - (1+\tau)K} \right)\). In summary,

Proposition 2: When \(V(w) - \frac{\tau}{1+\tau} w\) is decreasing over \([0, +\infty)\), the optimal contract is characterized as follows

\[
\begin{cases}
x = 1 \\
\tau = w - \frac{1+r}{r} \frac{pB}{\Delta p} + \frac{B}{\Delta p} \\
\bar{w} = (1+r) \left( \frac{w}{x} + \frac{(1-p)B}{\Delta p} \right)
\end{cases}
\]
\[ x = 1 \]

- if \( w \geq \frac{1+r}{r} \frac{pB}{\Delta p} \),

\[ v = w - \frac{1+r}{r} \frac{pB}{\Delta p} + \frac{B}{\Delta p} \]

\[ c = w - \frac{1+r}{r} \frac{pB}{\Delta p} \]

\[ w = w \]

- If \( 0 \leq w < \frac{1+r}{r} \frac{pB}{\Delta p} \),

\[ x = \inf \left( 1, \frac{w}{\frac{pB}{\Delta p} - (1+r)K} \right) \]

\[ w = (1+r) \left( \frac{w}{x} + \frac{(1-p)B}{\Delta p} \right) \]

\[ c = 0 \]

\[ w = (1+r) \max \left( 0, \frac{w}{x} - \frac{pB}{\Delta p} \right) \]

\[ c = \frac{1}{1+r} \left( \frac{w}{x} - \frac{pB}{\Delta p} \right) - \frac{1}{1+r} \max \left( 0, \frac{w}{x} - \frac{pB}{\Delta p} \right) \]

It is obvious that this result corresponds to the second regime presented in the proposition 1.

6 Conclusion

In this paper we look at the choice between the liquidation and the recapitalization by incorporating the recapitalization possibility into the literature on optimal long term financial contracting in the presence of moral hazard problems. We find that these two options are resorted to after a poor performance when the entrepreneur’s share to the future cash flows is low enough. Two regimes are possible in solving for the financial distress situation.

Our analysis leaves aside some interesting questions such as what kind of financial instruments will be issued during the reorganization period? What determines the type of new instruments issued? Whether there exists a need for having a formal bankruptcy procedure. We intend to address these questions in further researches.

A Appendix 1: Derivation of continuation contract for the second period

* If \( w_2 \geq \frac{pB}{\Delta p} \): one can easily verify that \( \begin{cases} x_2 = 1 \\ \overline{v}_2 = w_2 + \frac{(1-p)B}{\Delta p} > 0 \end{cases} \) satisfy all constraints.

\( \overline{c}_2 = w_2 - \frac{pB}{\Delta p} \geq 0 \)

Because \( \arg \max x_2 = 1 \) and \( \arg \max U(c) - c = [0, +\infty) \), it is also solution and \( V_2(w_2) = \overline{R} \)

* If \( 0 \leq w_2 < \frac{pB}{\Delta p} \)

Note that inducing high effort requires the creation of a non negative wedge between \( \overline{v}_2 \) and \( \overline{c}_2 \). Moreover, \( w_2 \) must be non negative. So, at the optimum, we can have either \( (\overline{v}_2, \overline{c}_2) \in R_+^2 \) or \( (\overline{v}_2, \overline{c}_2) \in R_+ \times [-K, 0) \) and \( V_2(w_2) = \max \left( V'_2(w_2); V''_2(w_2) \right) \) where
$V'_2(w_2)$ is defined for $(\tau_2, c_2) \in R^2_+$ and $V''_2(w_2)$ defined for $(\tau_2, c_2) \in R_+ \times [-K, 0)$.

$$V'_2(w_2) = \text{Max } x_2 \tilde{R}$$

for all $w_2 \geq 0$ s.t

$$x_2(p\tau_2 + (1 - p)c_2) = w_2 \tag{21}$$

$$\tau_2 - c_2 \geq \frac{B}{\Delta p} \tag{22}$$

$$(x_2, \tau_2, c_2) \in [0, 1] \times R^2_+ \tag{23}$$

From (21) & (22), we obtain, $x_2 \leq \frac{w_2}{\Delta p} < 1$. So, $0 \leq x_2 \leq \frac{w_2}{\Delta p}$. Hence, we have solution

$$\begin{align*}
\begin{cases}
x_2 = \frac{w_2}{\Delta p} \\
\tau_2 = \frac{B}{\Delta p} \\
c_2 = 0
\end{cases}
\end{align*}$$

and $V'_2(w_2) = \frac{R}{\Delta p} w_2$, 

$$\begin{align*}
V''_2(w_2) = \text{Max } x_2 [\tilde{R} + (1 - p) \tau c_2]
\end{align*}$$

for all $w_2 \geq 0$ s.t.

$$x_2 (p\tau_2 + (1 - p)(1 + \tau)c_2) = w_2 \tag{24}$$

$$\tau_2 - (1 + \tau)c_2 \geq \frac{B}{\Delta p} \tag{25}$$

$$(x_2, \tau_2, c_2) \in [0, 1] \times R_+ \times [-K, 0) \tag{26}$$

Let $\tau_2 - (1 + \tau)c_2 = \frac{B}{\Delta p} + \varepsilon$ where $\varepsilon \geq 0$. From (24), we obtain $c_2 = \frac{1}{1 + \tau} \left( \frac{w_2}{\Delta p} - \frac{pB}{\Delta p} - p\varepsilon \right) \in [-K, 0) \Rightarrow \frac{w_2}{\Delta p} + p\varepsilon < x_2 \leq \frac{w_2}{\Delta p - (1 + \tau)K + p\varepsilon}$. Hence,

$$V''_2(w_2) = \text{Max } x_2 \left[ \tilde{R} - (1 - p) \frac{\tau}{1 + \tau} \left( \frac{pB}{\Delta p} + p\varepsilon \right) \right] + (1 - p) \frac{\tau}{1 + \tau} w_2$$

for all $w_2 \geq 0$ s.t.

$$\frac{w_2}{\frac{pB}{\Delta p} + p\varepsilon} < x_2 \leq \inf \left( 1, \frac{w_2}{\Delta p - (1 + \tau)K + p\varepsilon} \right)$$

Since $\tilde{R} \geq (1 - p) \frac{\tau}{1 + \tau} \left( \frac{pB}{\Delta p} + p\varepsilon \right) \geq 0$. Therefore, $x_2 \left[ \tilde{R} - (1 - p) \frac{\tau}{1 + \tau} \left( \frac{pB}{\Delta p} + p\varepsilon \right) \right] \leq \inf \left( 1, \frac{w_2}{\Delta p - (1 + \tau)K + p\varepsilon} \right) \times \left[ \tilde{R} - (1 - p) \frac{\tau}{1 + \tau} \left( \frac{pB}{\Delta p} + p\varepsilon \right) \right]$. Since the righ-hand side of this inequality is non-increasing with respect to $\varepsilon$, at the optimum we have $\varepsilon = 0$ i.e. $\tau_2 - (1 + \tau)c_2 = \frac{B}{\Delta p}$
Moreover, because there exists a solution for all constraints (11) that contradicts (contradiction). So, at the optimum

\[ V_2''(w_2) = \max_{x_2} x_2 \left[ \bar{R} - (1 - p) \frac{\tau + pB}{1 + \tau} \right] + (1 - p) \frac{\tau}{1 + \tau} w_2 \]

for all \( w_2 \geq 0 \) s.t.

\[
\frac{w_2}{\frac{p}{\Delta p}} < x_2 \leq \inf \left( 1, \frac{w_2}{\frac{p}{\Delta p} - (1 + \tau)K} \right)
\]

Thus,

For \( \frac{p}{\Delta p} = (1 + \tau)K < w_2 < \frac{p}{\Delta p} \): \( x_2 = 1 \) and \( V_2''(w_2) = (1 - p) \frac{\tau}{1 + \tau} w_2 + \bar{R} - (1 - p) \frac{\tau}{1 + \tau} \frac{p}{\Delta p} \).

For \( 0 \leq w_2 < \frac{p}{\Delta p} - (1 + \tau)K \): \( x_2 = \frac{w_2}{\frac{p}{\Delta p} - (1 + \tau)K} \) and \( V_2''(w_2) = \frac{\bar{R} - (1 - p)\tau K}{\frac{p}{\Delta p} - (1 + \tau)K} w_2 \).

Finally, \( V_2(w_2) = \begin{cases} 
\bar{R} & \text{for } w_2 \geq \frac{p}{\Delta p} \\
(1 - p) \frac{\tau}{1 + \tau} w_2 + \bar{R} - (1 - p) \frac{\tau}{1 + \tau} \frac{p}{\Delta p} & \text{for } \frac{p}{\Delta p} - (1 + \tau)K \leq w_2 < \frac{p}{\Delta p} \\
\frac{\bar{R} - (1 - p)\tau K}{\frac{p}{\Delta p} - (1 + \tau)K} w_2 & \text{for } 0 \leq w_2 < \frac{p}{\Delta p} - (1 + \tau)K
\end{cases} \)

### B Appendix 2: Proof of lemma 2

To start with, we demonstrate that at the optimum, \( \xi_1 \) is non positive. Indeed, suppose that there exists a solution \((\bar{c}_1, \xi_1, \bar{w}_2, w_2)\) such that \( \xi_1 > 0 \). Since \( w_1 < \frac{p}{\Delta p} \left( 1 + \frac{1}{1 + \tau} \right) \), if \( \xi_1 > 0 \), then from constraints (11) – (12), one has \( \bar{w}_2 \) must be strictly less than \( \frac{p}{\Delta p} \).

Moreover, because \( \xi_1 \) is strictly greater than 0, there is a \( \varepsilon > 0 \) so that \( \xi_1 - \varepsilon \geq 0 \). Define \( \xi'_1 = \xi_1 - \varepsilon \) and \( \bar{w}_2' = \bar{w}_2 + (1 + r)\varepsilon \). It is easy to check that \((\bar{c}_1', \xi'_1, \bar{w}_2', w_2')\) satisfies all constraints (11) – (14). Since \( V_2 \) is strictly increasing over \( \left[ 0, \frac{p}{\Delta p} \right) \) and constant over \( \left( \frac{p}{\Delta p}, +\infty \right) \), we have\(^5\)

\[
\bar{R} + p \left[ U(\bar{c}_1) - \bar{c}_1 \right] + (1 - p) \left[ U(\xi_1) - \xi_1 \right] + \frac{pV_2(\bar{w}_2) + (1 - p)V_2(w_2)}{1 + r}
\]

\[
< \bar{R} + p \left[ U(\bar{c}_1) - \bar{c}_1 \right] + (1 - p) \left[ U(\xi_1) - \xi_1 \right] + \frac{pV_2(\bar{w}_2') + (1 - p)V_2(w_2) + (1 + r)\varepsilon}{1 + r}
\]

(contradiction). So, at the optimum \( \xi_1 \leq 0 \).

Next, we prove that \( \bar{c}_1 \) is non positive. Similarly, assume that there exists a solution \((\bar{c}_1', \xi_1, \bar{w}_2, w_2)\) where \( \bar{c}_1 = \bar{c}_1 - \varepsilon \geq 0 \) and \( \bar{w}_2 = \bar{w}_2 + (1 + r)\varepsilon \) with \( \varepsilon > 0 \). One can see that \((\bar{c}_1', \xi_1, \bar{w}_2', w_2')\) satisfies all constraints and

\[
\bar{R} + p \left[ U(\bar{c}_1) - \bar{c}_1 \right] + (1 - p) \left[ U(\xi_1) - \xi_1 \right] + \frac{pV_2(\bar{w}_2) + (1 - p)V_2(w_2)}{1 + r}
\]

\[
\leq \bar{R} + p \left[ U(\bar{c}_1') - \bar{c}_1' \right] + (1 - p) \left[ U(\xi_1) - \xi_1 \right] + \frac{pV_2(\bar{w}_2 + (1 + r)\varepsilon) + (1 - p)V_2(w_2)}{1 + r}
\]

\(^5\)Note that when \( c \geq 0, U(c) - c = 0 \).
(contradiction). Thus, it is better to set $r_1 \leq 0$ at the optimum.

Finally, we need to show that the incentive compatibility constraint is binding. The Lagrangian for this problem is written as:

$$L(\xi_1, \xi_1, \bar{w}, \bar{w}, \lambda_1, \lambda_2) = \bar{R} + p [U(\xi_1) - \xi_1] + (1 - p) [U(\xi_1) - \xi_1] + \frac{pV_2(\bar{w}) + (1 - p)V_2(\bar{w})}{1 + r}$$

$$+ \lambda_1 \left( w_1 - pU(\xi_1) - (1 - p)U(\xi_1) - \frac{p\bar{w}_2 + (1 - p)\bar{w}_2}{1 + r} \right)$$

$$+ \lambda_2 \left( U(\xi_1) - U(\xi_1) + \frac{\bar{w}_2 - \bar{w}_2}{1 + r} - \frac{B}{\Delta p} \right)$$

FOC for $\xi_1$ and $\xi_1$ implies that $\lambda_1 = \frac{\dot{r}}{1 + r}$. If $\lambda_2 = 0$, then FOC for $\bar{w}$ and $\bar{w}$ results in $V_2(\bar{w}) = V_2(\bar{w}) = \frac{\dot{r}}{1 + r}$ (contradiction). Therefore, $\lambda_2$ must be strictly positive, i.e. the ICC is binding.

### Appendix 3: Proof of lemma 3

#### C.1 There exists an unique, continuous and bounded solution $V$?

Let $C_b(R_+)$ be bounded continuous function space and define an Bellman operator $T$ as follows

$$T : v(w) \rightarrow Tv(w)$$

where

$$Tv(w) = \max_{x, \xi, \xi, \bar{w}, \bar{w}} \left[ \bar{R} + p (U(\xi) - \xi) + (1 - p)(U(\xi) - \xi) + \frac{p\bar{w} + (1 - p)v(w)}{1 + r} \right]$$

for all $w \geq 0$ s.t.

$$x [pU(\xi) + (1 - p)U(\xi) + \frac{p\bar{w} + (1 - p)w}{1 + r}] = w$$

$$U(\xi) - U(\xi) + \frac{\bar{w} - w}{1 + r} \geq \frac{B}{\Delta p}$$

$$\lambda_2 \left( U(\xi) - U(\xi) + \frac{\bar{w} - w}{1 + r} \right)$$

$$\lambda_2 \left( U(\xi) - U(\xi) + \frac{\bar{w} - w}{1 + r} \right) \geq \frac{B}{\Delta p}$$

$$\left( \xi, \xi, \bar{w}, \bar{w} \right) \in [-K, +\infty)^2 \times R_+^2$$

$$x \in [0, 1]$$

To prove that there exists an unique, continuous and bounded function $V$, we need to show that $T$ maps $C_b(R_+)$ to itself and that $T$ is contraction. Indeed, because $U(c) - c$ and $v(w)$ are all bounded, $Tv(w)$ is also bounded.

Let $\dot{w} \geq 0$ is the smallest point at which $v(w)$ reaches its maximum. For any value of $w \in \left[ 0; \frac{pB}{\Delta p} + \frac{\dot{w}}{1 + r} \right]$, there is no loss of generality to restrict $(x, \xi, \xi, \bar{w}, \bar{w})$ in $[0, 1] \times [-K, \frac{B}{\Delta p}] \times [-K, 0] \times [0, \dot{w}]^2$. Since the objective function is continuous and the set $[0, 1] \times [-K, \frac{B}{\Delta p}] \times [-K, 0] \times [0, \dot{w}]^2$ is compact, from the Maximum theorem, the function $Tv(w)$ is continuous on the interval $\left[ 0; \frac{pB}{\Delta p} + \frac{\dot{w}}{1 + r} \right]$. When $w > \frac{pB}{\Delta p} + \frac{\dot{w}}{1 + r}$, $(x, \xi, \xi, \bar{w}, \bar{w}) = \ldots$
\((1, \frac{1}{r} (w - \frac{\dot{w}}{1+r}), 0, \dot{w}, \ddot{w})\) is solution, i.e. for \(w > \frac{p^B}{\Delta p} + \frac{\dot{w}}{1+r} \), \(T_v(w) = R + \frac{v(\dot{w})}{1+r} = T_v\left(\frac{p^B}{\Delta p} + \frac{\dot{w}}{1+r}\right)\). So, \(T_v(w)\) is continuous on \([0, +\infty)\). \(T_v(w)\) are thus both bounded and continuous. This means that the operator \(T\) maps \(C_b(R_+)\) to itself.

We can easily verify that the operator \(T\) satisfies two Blackwell’s sufficient conditions for a contraction (monotonicity and discounting). It implies that \(T\) is contraction. Hence, by the Contraction Mapping Theorem, \(T\) has an unique fixed point \(V \in C_b(R_+)\).

### C.2 \(V\) is non decreasing?

Take \(w\) and \(w'\) such that \(w' > w \geq 0\).

Let \((x, \bar{c}, \zeta, \bar{w}, w)\) be solution in the program that defines \(T_v(w)\).

Take \((x, \bar{c}', \zeta, \bar{w}, w)\) such that \(p \left( U(\bar{c}') - U(\bar{c}) \right) = \frac{w' - w}{x} > 0\). Since \(U(c)\) is increasing function, we have \(\bar{c}' > \bar{c} \geq -K\). Moreover,

\[
\begin{align*}
    x \left( pU(\bar{c}') + (1 - p)U(\bar{c}) + \frac{p\bar{w} + (1 - p)w}{1 + r} \right) &= x \left( \frac{w' - w}{x} + \frac{w}{x} \right) = w' \\
    U(\bar{c}') - U(\bar{c}) + \frac{\bar{w} - w}{1 + r} &> U(\bar{c}) - U(\bar{c}) + \frac{\bar{w} - w}{1 + r} \geq \frac{B}{\Delta p}
\end{align*}
\]

Thus, \((x, \bar{c}', \zeta, \bar{w}, w)\) is feasible choice in the program that defines \(T_v(w')\)

\[
T_v(w') \geq x \left[ R + p \left( U(\bar{c}') - \bar{c}' \right) + (1 - p) \left( U(\bar{c}) - \bar{c} \right) + \frac{pv(\bar{w}) + (1 - p)v(w)}{1 + r} \right]
\]

\[
\geq x \left[ R + p \left( U(\bar{c}) - \bar{c} \right) + (1 - p) \left( U(\bar{c}) - \bar{c} \right) + \frac{pv(\bar{w}) + (1 - p)v(w)}{1 + r} \right] = T_v(w)
\]

i.e. \(T_v(w)\) is non decreasing function.

### C.3 \(V\) is concave?

Consider

\[
T_v^c(w) = \max_{x, \zeta, \bar{w}, w} \left[ R + p \left( U(\bar{c}) - \bar{c} \right) + (1 - p) \left( U(\bar{c}) - \bar{c} \right) + \frac{pv(\bar{w}) + (1 - p)v(w)}{1 + r} \right]
\]

for all \(w \geq \frac{p^B}{\Delta p} - (1 + \tau)K\) s.t

\[
\begin{align*}
    pU(\bar{c}) + (1 - p)U(\bar{c}) + \frac{p\bar{w} + (1 - p)w}{1 + r} &= w \\
    U(\bar{c}) - U(\bar{c}) + \frac{\bar{w} - w}{1 + r} &\geq \frac{B}{\Delta p} \\
    (\bar{c}, \zeta, \bar{w}, w) &\in [-K, +\infty)^2 \times R_+^2
\end{align*}
\]

So,

\[
T_v(w) = \max_{x, w^c} xT_v^c(w^c)
\]
for all $w \geq 0$ s.t.

\[
x w^e = w
\]

\[
x \in [0, 1]
\]

\[
w^e \in \left[ \frac{pB}{\Delta p} - (1 + \tau)K, +\infty \right]
\]

First, we will prove that if $v(w)$ is concave, then $T^c v(w)$ is concave. Indeed, let $(\overline{\epsilon}, \overline{\epsilon'}, \overline{w}, \overline{w'})$ be solution for $T^c v(w)$ and $(\overline{\epsilon'}, \overline{\epsilon'}, \overline{w'}, \overline{w'})$ for $T^c v(w')$. Define $w_\lambda = \lambda w + (1 - \lambda)w'$. Define $(\overline{\epsilon}_{\lambda}, c_\lambda, \overline{w}_{\lambda}, \overline{w}_{\lambda})$ such that

\[
\overline{w}_\lambda = \lambda \overline{w} + (1 - \lambda)\overline{w'}
\]

\[
\overline{w}_\lambda = \lambda w + (1 - \lambda)w'
\]

\[
U(\overline{\epsilon}_\lambda) = \lambda U(\overline{\epsilon}) + (1 - \lambda)U(\overline{\epsilon'})
\]

\[
U(c_\lambda) = \lambda U(\overline{\epsilon}) + (1 - \lambda)U(\overline{\epsilon'})
\]

Hence, $(\overline{\epsilon}_{\lambda}, c_\lambda, \overline{w}_{\lambda}, \overline{w}_{\lambda}) \in [-K, +\infty) \times R^2$ and $U(\overline{\epsilon}_{\lambda}) - U(c_\lambda) + \frac{\overline{w}_\lambda - \overline{w}_\lambda}{1 + r} \geq \frac{R}{\Delta p}$. It implies that $(\overline{\epsilon}_{\lambda}, c_\lambda, \overline{w}_{\lambda}, \overline{w}_{\lambda})$ is feasible choice for $T^c v(w_\lambda)$:

\[
T^c v(w_\lambda) \geq R + p \left( U(\overline{\epsilon}_{\lambda}) - U(c_\lambda) \right) + \frac{(1 - p) \left( U(\overline{\epsilon}_{\lambda}) - c_\lambda \right) + pv(\overline{w}_{\lambda}) + (1 - p)v(w_\lambda)}{1 + r}
\]

The concavity of function $v(w)$ implies

\[
v(\overline{w}_\lambda) \geq \lambda v(\overline{w}) + (1 - \lambda) v(\overline{w'})
\]

\[
v(w_\lambda) \geq \lambda v(w) + (1 - \lambda) v(w')
\]

Since $U$ is concave and increasing function, from definition of $\overline{\epsilon}_{\lambda}$, we have $\overline{\epsilon}_{\lambda} \leq \lambda \overline{\epsilon} + (1 - \lambda)\overline{\epsilon'}$ and thus, $U(\overline{\epsilon}_{\lambda}) - U(\overline{\epsilon}) \geq \lambda [U(\overline{\epsilon}) - \overline{\epsilon}] + (1 - \lambda) \left[ U(\overline{\epsilon'}) - \overline{\epsilon'} \right]$. Similarly, $U(c_\lambda) - c_\lambda \geq \lambda [U(\overline{\epsilon}) - \overline{\epsilon}] + (1 - \lambda) \left[ U(\overline{\epsilon'}) - \overline{\epsilon'} \right]$. Finally, we get

\[
T^c v(w_\lambda) \geq \lambda T^c v(w) + (1 - \lambda) T^c v(w')
\]

i.e. $T^c v(w)$ is concave.

Now, we establish the concavity of $T v(w)$.

we have

\[
Tv(w) = Max \ w \frac{T^c v(w^c)}{w^c}
\]

for all $w \geq 0$ s.t

\[
w^c \geq \max(w, \frac{pB}{\Delta p} - (1 + \tau)K)
\]

Define $\tilde{w}^*$ by $\tilde{w}^* = \frac{\tilde{w}}{1 + r} + \frac{pB}{\Delta p}$.

Similarly to the previous parts, one can verify that $T^c v(w)$ is continuous on its domain and constant over $[\tilde{w}^*, +\infty)$. Therefore, the mapping $w^c \rightarrow \frac{T^c v(w^c)}{w^c}$ reaches its maximum in $[\frac{pB}{\Delta p} - (1 + \tau)K, \tilde{w}^*].$ Let $\arg \max \frac{T^c v(w^c)}{w^c} = [w_\lambda^c, \tilde{w}^c]$, possibly reduced to a point $\Rightarrow$
\[ w^c \geq \frac{pB}{\Delta p} - (1 + \tau)K. \]

\[
\text{arg max}_{w^c \geq \max \left( w, \frac{pB}{\Delta p} - (1 + \tau)K \right)} \frac{T^c(w^c)}{w^c}
\begin{cases}
    w & \text{if } w \geq \bar{w}^c \\
    [w, \bar{w}^c] & \text{if } \bar{w}^c > w \geq w^c \\
    [w^c, \bar{w}^c] & \text{if } w^c > w \geq 0
\end{cases}
\]

so,

\[ T^c(v(w)) = \begin{cases}
    T^c(v(w)) & \text{if } w \geq \bar{w}^c \\
    w \times A & \text{if } 0 \leq w < \bar{w}^c
\end{cases} \quad \text{where } A \text{ does not depend on } w
\]

\[ \Rightarrow T^c(v(w)) \text{ is concave.} \]

**C.4** \( V \) is constant over \( \left[ \frac{1 + \tau}{r} \frac{pB}{\Delta p} + \infty, +\infty \right) \) and strictly increasing over \( \left[ 0, \frac{1 + \tau}{r} \frac{pB}{\Delta p} \right) \)

From previous parts, we know

\[ V(w) = \tilde{R} + \frac{V(\hat{w})}{1 + r} \text{ if } w \geq \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \]

where \( \hat{w} \) is smallest point at which \( V(w) \) reaches its maximum.

Take \( w_1 < w_2 < \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \). Because \( V(w) \) is non decreasing, \( V(w_1) \leq V(w_2) \leq V \left( \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \right) \). If \( V(w_1) = V(w_2) \), then with the concavity of \( V(w) \), we get \( V(w_1) = V(w_2) = V \left( \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \right) \). It implies that \((x, \bar{c}, \bar{c}, w) = \left( 1, \frac{pB}{\Delta p}, 0, \hat{w}, \bar{w} \right)\) is solution to the program that defines \( V(w_1) \) and so, \( w_1 = \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \) (contradiction). Hence, \( V(w) \) is strictly increasing over \( \left[ 0, \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \right) \). Moreover, it is constant over \( \left[ \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r}, +\infty \right) \), that means \( \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \) is the smallest point of \( \text{arg max} \ V(w) \Rightarrow \frac{pB}{\Delta p} + \frac{\hat{w}}{1 + r} \Rightarrow \hat{w} = \frac{1 + \tau}{r} \frac{pB}{\Delta p} \).

**References**


