Prompt Corrective Action with Recapitalization
Possibility in a Continuous - Time Model

VO Thi Quynh Anh*
GREMAQ, Toulouse School of Economics

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Abstract

Our paper addresses the optimality of the Prompt Corrective Action - type regulation by adapting the dynamic model of entrepreneurial finance to banking regulation. In a dynamic moral hazard setting, we first derive the optimal allocation between the banker and the regulator and then implement it by a menu of regulatory tools. Our main findings are the following: first, the insurance premium is risk-based premium where the risk is measured by the amount of capital; second, our model-implied capital regulation share several similarities with the US PCA. According to our capital regulation, regulatory supervision should be realized in the spirit of gradual intervention and the book-value of capital is used as information to trigger intervention. Banks with high capital are not subject to any restrictions. Dividend distribution is prohibited in banks with intermediate level of capital. When banks have low capital level, a plan of recapitalization is required and in the worse case, banks are placed in liquidation. Banks should be forced into resolution when they still have positive capital.

Key words: Prompt Corrective Action, Capital Regulation, Dynamic contracting, Recapitalization.

JEL Codes: D82, G21, G28

1 Introduction

Following the implementation of the first Basel Accord (1988), academic research has spent a lot of effort in assessing the effects of minimum capital requirement on excessive risk taking incentives. A conclusion derived from these works is that imposing minimum regulatory capital requirements itself does not constitute an adequate solution for reducing excessive risk taking, particularly in today’s world, financial innovation has produced new

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markets and instruments that make it easy for banks and their employees to make huge bets easily and quickly. This thinking drives the Basel Committee to incorporate in Basel Accord II the pillar 2 - supervisory review - and the pillar 3 - market discipline - as complementary to the pillar 1 - minimum capital requirement. The Basel Committee states that the goal of the pillar 2 is to enable early supervisory intervention if the capital does not provide a sufficient buffer against risk. However, it remains silent on the way to implement this principle in practice, or in other words, it remains silent on the threshold and forms of intervention.

In the US, a system of predetermined capital/asset ratios that trigger structured actions by supervisor, which is called as Prompt Corrective Action (PCA), was introduced in the 1991 Federal Deposit Insurance Corporation Improvement Act (FDICIA). PCA classifies banks in five categories depending on capital ratios: well capitalized, adequately capitalized, undercapitalized, significantly undercapitalized and critically undercapitalized. Imposition of regulatory restraints on banks becomes more and more severe the lower their capital ratios. For instance, well capitalized and adequately capitalized banks face no restrictions. Undercapitalized banks don’t have right to capital distribution (dividend or stock repurchase). Significantly undercapitalized banks must submit a recapitalization plan. Critically undercapitalized banks have to be placed in receivership within 90 days. Some positive observed effects of FDICIA in creating the appropriate incentives for banks, the deposit insurer and the prudential supervisor result in the increasing number of recommendations to introduce PCA-type provisions in other countries. Over the past years, Japan, Korea and Mexico have adopted a similar system of the US PCA. Recently, the European Shadow Financial Regulatory Committee (ESFRC) made a proposal aimed at dealing with problem banks. One of the recommendations in their proposal is to implement a PCA regime in each individual Member State. In such circumstances, a rigorous study of the optimality of PCA-type regulation seems timely and relevant.

Our paper will address this issue by adapting the dynamic model of entrepreneurial finance to banking regulation. We consider an infinitely repeated relationship between the banker and the Deposit Insurance Corporation (DIC) which is subject to moral hazard problem. The banker runs a bank whose profitability depends on her effort. High effort improves the expected cash-flows of the bank. However, exerting high effort generates a loss of private benefits to the banker. The DIC offers the deposit insurance services and supervises the banker on behalf of the depositors. To provide the banker with appropriate incentives, the DIC can control her compensations, require her to inject more capital to the bank or force her to liquidate it. In this framework, we first characterize the optimal allocation of resources between the banker and the DIC. The method we use to solve for the optimal allocation is the dynamic programming technique. Specifically, we use the banker’s expected discounted utility as state variable and the optimal allocation will be contingent on it. After the characterization, we construct a regulatory menu that can implement the optimal allocation. Our menu includes three instruments: bank chartering, capital regulation and deposit insurance premium. Bank chartering determines the condition to set up a bank. Deposit insurance premium defines the payments paid to the DIC at every period. Capital regulation is characterized by the regulatory restrictions
on dividend distribution, recapitalization plan and liquidation. Our main findings are the following: first, the insurance premium is risk-based premium where the risk is measured by the amount of capital; second, our model-implied capital regulation shares several similarities with the US PCA. According to our capital regulation, regulatory supervision should be realized in the spirit of gradual intervention and the book-value of capital is used as information to trigger intervention. Banks with high capital are not subject to any restrictions. Dividend distribution is prohibited in banks with intermediate level of capital. When banks have low capital level, a plan of recapitalization is required and in the worse case, banks are placed in liquidation.

Recently, there is a growing litterature analyzing dynamic moral hazard. It typically consists of DeMarzo and Fishman (2004); DeMarzo and Sannikov (2006); Sannikov (2006); Blais, Mariotti, Plantin and Rochet (2006) (BMPR (2006)); Blais, Mariotti, Rochet and Villeneuve (2007) (BMRV(2007)) and DeMarzo and Sannikov (2007). In general, these papers study the optimal dynamic contract in a setting in which a risk neutral entrepreneur seeks funding from risk neutral investors to finance a project that pays stochastic cash-flows over many periods. Their contracting relationship is subject to moral hazard problem coming either from the unobservability of cash-flows or from the hidden effort. The entrepreneur is liable for payments to the investors only to the extent of current revenues. In addition to some variations in modelling, these papers propose different methods to implement the optimal contract and so, generate different insights. For example, to get interesting implications for an optimal capital structure, DeMarzo and Sannikov (2006) consider to implement the optimal contract by a combination of equity, long-term debt and credit line. Having the same objective but BMPR (2006) study an implementation realized via debt, equity and cash reserves. By implementing the optimal contract through the firm’s payout policy, DeMarzo and Sannikov (2007) provides an explanation for the smoothness of corporate dividends relative to earnings and cash-flows.

Our paper is also based on a dynamic moral hazard model. However, compared to the above papers, we relax the limited liability of the agent (the banker) and so, allow the principal (the DIC) to require the banker to inject money during their relationship. Moreover, in this paper, we consider the implementation through a menu of regulatory instruments available in practice of banking regulation. Therefore, we is able to discuss the issue of optimality of PCA-type regulation.

The litterature on PCA is mainly empirical. Since the introduction of the US PCA, there have been several attempts to assess its functioning. Some papers recognized significant impacts of PCA in terms of raising capital ratios and reducing risk for banks. Nevertheless, Barth et al. (2004) in a study of bank regulation and supervision in 107 countries raise doubts about government policies that rely excessively on direct government regulation and supervision of banks. For the theoretical analysis of PCA, we can cite Shim (2006), Freixas and Parigi (2007). The most relevant for our work is Shim (2006). Our paper takes the same approach as its, i.e. applying the dynamic moral hazard model of entrepreneurial finance to banking regulation. However, this paper uses the discrete-time model and does not account for the possibility of recapitalization by the banker. Our paper can be seen as its extension since we take into consideration the costly recapitaliza-
tion option. Due to the advantage in tractability of a continuous-time model, we succeed to fully characterize the optimal allocation in the extended setting.

The paper proceeds as follows. In section 2, we present the model in a continuous-time setting. Section 3 is devoted to the characterization of the optimal allocation of resources. Section 4 shows how the optimal allocation is implemented by regulatory instruments. Finally, section 5 concludes.

2 Model

We consider here a repeated relationship between a risk-neutral banker who wants to operate a bank and the risk-neutral Deposit Insurance Corporation (DIC) who is in charge of insuring the deposits and supervising the bank.

More specifically, at the initial time, the banker has an endowment of cash $E$. If she transfers $E_0$ to the DIC, she can set up a bank, collect $D$ units of deposits and invests them in a long-term risky loan portfolio. The cumulated cashflows of this portfolio evolve according to the following diffusion process

$$dR_t = \mu A_t dt + \sigma Z^A_t$$

where $\mu$, $\sigma$ are positive constants; $A_t$ is effort level of the banker at time $t$ and $Z^A = \{Z^A_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is standard Brownian motion defined on the measurable space $(\Omega, \mathcal{F})$ equipped with a probability measure $P^A$ induced by an effort process $A = \{A_t, 0 \leq t < \infty\}$. For simplicity, we assume that the set of feasible effort levels contains two elements $\{0, 1\}$. Effort is costly for the banker in the sense that she enjoys a private benefit $B$ if exerting low effort ($A_t = 0$). Denote by $v(A_t)$ the banker’s benefits associated with effort level $A_t$. Hence, $v(0) = B$ and $v(1) = 0$. We assume that $B < \mu$, i.e. exerting high effort is efficient.

In our model, the moral hazard problem comes from the unobservability of the banker’s effort. That means, whereas the cashflows process $R = \{R_t, 0 \leq t < \infty\}$ is publicly observable by both the DIC and the banker, the effort level $A_t$ is private information of the latter. An allocation of resources between the banker and the DIC specifies, based on the entire histories of cashflows realizations, a liquidation time $\tau(R_s, 0 \leq s < \tau)$ and a flow of payments $c = \{c_t(R_s, 0 \leq s \leq t), 0 \leq t < \tau\}$ to the banker at each time $t$ before the liquidation time. At any time, the bank can also be closed if the banker decides not to run the bank any more and switches to other businesses whose the best gives her an expected utility $\tilde{W}$. We assume that the value of the loan portfolio at the time of termination is zero.

Regarding the payments to the banker, we consider two alternatives. First, we assume that the banker cannot inject capital to the bank during its operation and so, the payments to the banker must be non-negative, i.e. $c_t \geq 0 \ \forall t$. Next, we will relax this limited liability constraint and assume that the DIC possesses an option of requiring the banker to contribute capital. We interpret this option as the recapitalization possibility and model it by allowing the lower bound of the banker’s payments to be a negative number$^1$.

$^1$Comments on the difference with Shim (2006)!!!
More concretely, we assume that for all $t$, $c_t$ must be greater or equal to $-K$ where $K > 0$. Furthermore, in our paper, the capital contribution is costly to the banker who bears a cost $\alpha$ for each unit of contributed capital. Therefore, the utility function of the banker becomes

$$U(c_t) = \begin{cases} c_t & \text{if } c_t \geq 0 \\ (1 + \alpha)c_t & \text{if } c_t \in [-K, 0) \end{cases} \quad (1)$$

If the banker discounts the future at the rate $\rho$ and the DIC at the riskless interest rate $r < \rho$, then, given an allocation $(\tau, c)$ and an effort strategy $A$, the total expected utility for the banker as of time 0, if she never quits, is given by

$$E^A \left[ \int_0^\tau e^{-\rho t} (U(c_t) + v(A_t)) \, dt + e^{-\rho \tau} \tilde{W} \right]$$

and for the DIC by

$$E^A \left[ \int_0^\tau e^{-rt} dR_t - \int_0^\tau e^{-rt} c_t dt \right] = E^A \left[ \int_0^\tau e^{-rt} (\mu A_t - c_t) dt \right]$$

where $E^A$ denotes the expectation under the probability measure $P^A$.

An effort strategy is defined as incentive compatible with respect to the allocation $(\tau, c)$ if it maximizes the total expected utility of the banker given $(\tau, c)$. Here we focus on the allocations that induce high effort every time and if facing these allocations, the banker will never choose to quit. We label such a class of allocations as incentive compatible one. The DIC’s problem is to find, among this class, the optimal allocation which provide him with highest payoff.

We denote by $P$ the probability measure generated by the effort process $A^* = \{A_t = 1 \ \forall 0 \leq t < \tau\}$ and by $E$ the expectation under $P$. Then, the DIC’s problem can be formulated as follows

$$\max_{\tau, c} E^\tau c \left[ \int_0^\tau e^{-rt} (\mu - c_t) dt \right]$$

subject to the following constraints

$$A^* = \{A_t = 1 \ \forall 0 \leq t < \tau\} \text{ is incentive compatible w.r.t } (\tau, c) \quad (3)$$

$$W_0 = E \left[ \int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} \tilde{W} \right] \quad (4)$$

$$E \left[ \left( \int_t^\tau e^{-\rho(s-t)} U(c_s) ds + e^{-\rho(\tau-t)} \tilde{W} \right) \bigg| \mathcal{F}_t \right] \geq \tilde{W} \quad \forall 0 \leq t < \tau$$

$$\forall 0 \leq t < \tau : \ c_t \in \Gamma - \text{an open subset of } \mathbb{R} \quad (5)$$

In line with DeMarzo and Sannikov (2006), by varying $W_0$, we can use this solution to consider different divisions of bargaining power between the banker and the DIC. For
example, if the DIC charters a banker from a competitive pool, then $W_0$ is chosen such that the DIC’s expected payoff as of time 0 is maximal subject to the constraint that the banker receives at least $\bar{W}$ (i.e. $W_0 \geq \bar{W}$).

3 Optimal allocation of resources

In this section, we present the derivation of the optimal allocation of resources. It will proceed in three steps. First, we state a result that relates the incentive compatibility condition of the effort process $A^*$ to the dynamic evolution of the banker’s continuation value. Next, we prove that the DIC’s payoff function can be determined as solution to an ordinary differential equation. Finally, by solving this equation, we find the optimal allocation.

3.1 Incentive compatibility condition

Here, we derive the incentive compatibility constraint for the banker relying on the martingale techniques introduced by Sannikov (2006).

Given an allocation $(\tau, c)$, for each $t < \tau$ denote by $W^A_t$ the banker’s continuation utility corresponding to an effort strategy $A = \{A_t, 0 \leq t < \tau\}$. It is the total expected utility the banker receives from the transfers from time $t$ on if she follows the strategy $A$.

$$W^A_t = E^A \left[ \left( \int_t^\tau e^{-\rho(s-t)} (U(c_s) + v(A_s)) \, ds + e^{-\rho(\tau-t)}\bar{W} \right) \bigg| \mathcal{F}_t \right]$$

The following lemma provides a useful representation of $W^A_t$.

**Lemma 1** There exists a stochastic process $G^A = \{G^A_t, 0 \leq t < \infty\}$ that represents the sensitivity of the banker’s continuation value to the cashflows, i.e.

$$dW^A_t = (\rho W^A_t - U(c_t) - v(A_t)) \, dt + \frac{G^A_t}{\sigma} \, (dR_t - \mu A_t \, dt) \tag{7}$$

**Proof.** Define by $V^A_t$ the lifetime utility of the banker under the allocation $(\tau, c)$ and an effort strategy $A = \{A_t, 0 \leq t < \tau\}$ conditionally on the information available at time $t < \tau$, then

$$V^A_t = E^A \left[ \left( \int_0^\tau e^{-\rho s} (U(c_s) + v(A_s)) \, ds + e^{-\rho \tau}\bar{W} \right) \bigg| \mathcal{F}_t \right]$$

So, we can rewrite $V^A_t$ as follows

$$V^A_t = \int_0^t e^{-\rho s} (U(c_s) + v(A_s)) \, ds + e^{-\rho t}W^A_t$$

that implies

$$dV^A_t = e^{-\rho t} \left[ U(c_t) + v(A_t) - \rho W^A_t \right] \, dt + e^{-\rho t}dW^A_t \tag{8}$$
On the other hand, by construction, we have that $V_t^A$ is $\mathcal{F}_t$-measurable and that for all $s \leq t < \tau$, $E^A [V_t^A | \mathcal{F}_s] = V_s^A$. So, $V_t^A$ is a $\mathcal{F}_t$-martingale. By the martingale representation theorem, there exists a progressively measurable stochastic process $G_t^A = \{G_t^A, \mathcal{F}_t, 0 \leq t < \infty\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and satisfying

$$E^A \left[ \int_0^t (e^{-\rho s} G_s^A)^2 \, ds \right] < \infty$$

for all $0 \leq t < \infty$ such that

$$V_t^A = V_0^A + \int_0^t e^{-\rho s} G_s^A dZ_s^A$$

(9) therefore,

$$dV_t^A = e^{-\rho t} G_t^A dZ_t^A$$

(10) (7) is automatically derived from (8) and (10). Q.E.D.

The lemma 1 provides a representation of the banker’s continuation utility as a diffusion process. This representation is valid for any effort strategy $A_t$. The question arised now is to determine the features concerning the drift and the volatility coefficients of the evolution of the banker’s promised value induced by the incentive - compatibility effort strategy.

Let $W_t$ and $Z_t$ be correspondently defined for the effort process $A_t^*$. Applying the lemma 1 to this process, we have

$$dW_t = (\rho W_t - U(c_t)) \, dt + G_t dZ_t$$

(11) where $G = \{G_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$. Obviously, at each time $t$, to decide what level of effort should be taken, the banker will rely on how this decision affects her continuation utility. Exerting high effort at time $t$ ($A_t = 1$) immediately causes a loss of private benefit $B$ to the banker but it improves her continuation value in expected term by $G_t^A$. Hence, intuitively, choosing high effort is profitable to the banker as long as $G_t^A \geq B$. A formal statement of this result is the following:

**Proposition 1** The strategy of exerting higt effort at any time is optimal for the banker if and only if the volatility of her continuation utility $G_t$ is at least equal to $\frac{B}{\mu}$ for all $t \in [0, \tau)$.

**Proof.** See appendix 1. ■

The proposition 1 means that for the incentive provision purpose, the banker has to bear some minimum risk, which is materialized by requiring that her continuation utility must be sensitive enough to the cashflows. The drift coefficient, in his turn, accounts for promise keeping. It states that $W_t$ should grow at the rate equal to the discount factor $\rho$ and fall due to the flow of repayments $U(c_t)$. 

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3.2 DIC’s continuation payoff function

The incentive compatibility condition being defined, it is the time for characterizing the DIC’s payoff function. Denote it by $F(W_t)^2$. It stands for the maximal continuation payoff the DIC can earn from all incentive compatible allocations if a continuation utility $W_t$ is promised to the banker at time $t$.

Consider an infinitesimal time interval $[t, t+dt)$, given a promised value $W_t$ for the banker at the beginning, if a transfer $c_t dt$ is paid to the banker during this period, the DIC immediately receives $dR_t + c_t dt$ and his continuation payoff is equal to $F(W_t + dW_t)$. Since the DIC discounts the future at the rate $r$, the term $rF(W_t)dt$ represents the expected change in his payoff during this time interval. Therefore, by construction, the following equality should be satisfied

$$rF(W_t)dt = Max_{c_t} \{ (\mu - c_t) + \frac{1}{dt}E[dF(W_t)] \}$$

equivalently,

$$rF(W_t) = Max_{c_t} \{ (\mu - c_t) + \frac{1}{dt}E[dF(W_t)] \}$$

To calculate the second term on the right-hand side of the equation (12), we need to find the dynamics of the DIC’s payoff. Using the Ito’s formula and based on the dynamic evolution of the banker’s continuation utility given by the equation (11), we have

$$dF(W_t) = \left( F'(W_t) (\rho W_t - U(c_t)) + \frac{1}{2} F''(W_t) G_t^2 \right) dt + F'(W_t) G_t dZ_t$$

Replacing $E[dF(W_t)] = \left( F'(W_t) (\rho W_t - U(c_t)) + \frac{1}{2} F''(W_t) G_t^2 \right) dt$ obtained from the equation (13) to the equation (12), we find that the DIC’s payoff function is solution to the following Hamilton-Jacobi-Bellman (HJB) equation

$$rF(W_t) = Max_{c_t \in \Gamma \cap G_t \geq \frac{\rho}{\mu} \sigma} \left[ (\mu - c_t) + F'(W_t) (\rho W_t - U(c_t)) + \frac{1}{2} F''(W_t) G_t^2 \right]$$

On the left-hand side of the equation (14), we have the expected instantaneous change in the DIC’s payoff. On the right-hand side, we have the sum of the expected instantaneous cashflows accruing to him and of the expected change in his continuation value. The maximization means that the current choice of $(c_t, G_t)$ is managed optimally.

We are now in position to consider the features of the optimal controls in (14). In the following, we will assume that the function $F(.)$ is concave, which will be checked later. This concavity implies that it is optimal to set $G_t$ at its minimal possible value $\frac{\rho}{\mu} \sigma$. Hence, because the expected discounted payoff of the DIC is concave with respect to the banker’s utility, reducing her exposure to risk is desirable for the DIC. The equation (14) is then rewritten as follows

$$rF(W_t) = F'(W_t) \rho W_t + \frac{1}{2} F''(W_t) \frac{B^2}{\mu^2} \sigma^2 + \mu + Max_{c_t \in \Gamma} \left[ -c_t - F'(W_t) U(c_t) \right]$$

Note that by the stationary property of the allocation, the DIC’s payoff function is common to all dates. Hence, we write it without any time label on the function symbol.
Concerning the compensation policy, we distinguish between two alternatives: with or without recapitalization possibility

### 3.2.1 Optimal allocation without recapitalization possibility

In case the banker cannot inject capital to the bank during its operation, the maximization in (15) is realized over the interval \([0, +\infty)\). The question raised here is to determine when a positive transfer should be paid to the banker. To provide the banker with the utility \(W_t\), the DIC has the option to pay a lump-sum transfer of \(c_t\) and switching to the allocation with promised utility \(W_t - c_t\). The DIC’s payoff corresponding to this compensation structure is equal to \(F(W_t - c_t) - c_t\). So, giving a positive compensation to the banker is optimal for the DIC if and only if

\[
F(W_t) \leq F(W_t - c_t) - c_t
\]

In other words, paying a positive transfer is optimal over the range of \(W_t\) where the function \(F(W_t) + W_t\) is nonincreasing. Define \(W^*\) by

\[
W^* = \inf \left\{ W : F'(W) = -1 \right\}
\]

Owing to the concavity of the function \(F\), we obtain

\[
c_t > 0 \text{ if and only if } W_t > W^*
\]

Since the banker has the possibility to quit and take an outside option with reservation utility \(\tilde{W}\), on the equilibrium path, \(W_t\) can not fall below \(\tilde{W}\) and the DIC must close the bank once \(W_t\) reaches this boundary value, which implies \(F(\tilde{W}) = 0\).

In summary, over the interval \([\tilde{W}, W^*]\), the function \(F\) is determined by the following ordinary differential equation

\[
rF(W_t) = F'(W_t)\rho W_t + \frac{1}{2} F''(W_t) \frac{B^2}{\mu^2} \sigma^2 + \mu \tag{16}
\]

There are already two boundary conditions associated with (16). The first one is \(F(\tilde{W}) = 0\). The second is the usual smooth-pasting condition \(F'(W^*) = -1\) which ensures that \(W^*\) is dividend threshold. However, since the value \(W^*\) itself is unknown, we need a supplementary third condition. This condition determines the optimality of \(W^*\), that is \(F''(W^*) = 0\).

**Proposition 2** The optimal allocation is characterized through the continuation utility \(W\) of the banker whose the dynamic is governed by the following stochastic differential equation

\[
dW = [\rho W - c(W)] dt + \frac{B}{\mu} \sigma dZ
\]
The DIC’s payoff is determined by the function $F$ - solution to

$$
\begin{cases}
  rF(W) = F'(W)\rho W + \frac{1}{2}F''(W)\frac{B^2}{\mu^2}\sigma^2 + \mu & \text{for } W \in [\tilde{W}, W^*] \\
  F(W) = F(W^*) - (W - W^*) & \text{for } W > W^*
\end{cases}
$$

with three boundary conditions: $F(\tilde{W}) = 0, F'(W^*) = -1$ and $F''(W^*) = 0$. The banker receives compensations according to the policy

$$c(W) = \max (W - W^*, 0)$$

The bank will be closed in the first time $W$ reaches $\tilde{W}$

$$\tau = \inf \left\{ t : W_t = \tilde{W} \right\}$$

**Proof.** See appendix 2

### 3.2.2 Optimal allocation with costly recapitalization possibility

In this subsection, we extend the previous one and assume that during the operation of the bank, the DIC has possibility to ask the banker to contribute capital. So, the maximization in (15) is realized over the interval $[-K, \infty)$ and the banker’s utility function is piecewise linear as (1).

With this extension, a complementary question need to be addressed, that is when the recapitalization occurs. Similarly to the arguments applied to the previous section, we compare the DIC’s payoff between different payment structures. Given the utility $W_t$ promised to the banker, if the DIC requires the banker to contributes $-c_t > 0$, because of the cost of capital contribution, he has to move to the allocation with promised utility $W_t - (1+\alpha)c_t$ and gets a payoff $F(W_t - (1+\alpha)c_t) - c_t$. Therefore, demanding a recapitalization is optimal if and only if

$$F(W_t) \leq F(W_t - (1+\alpha)c_t) - c_t$$

or

$$F(W_t) + \frac{W_t}{1+\alpha} \leq F(W_t - (1+\alpha)c_t) + \frac{W_t - (1+\alpha)c_t}{1+\alpha}$$

A recapitalization should happen over the range of $W_t$ where the function $F(W_t) + \frac{W_t}{1+\alpha}$ is nondecreasing. Define $\tilde{W}$ by

$$\tilde{W} = \inf \left\{ W : F'(W) = -\frac{1}{1+\alpha} \right\}$$

Hence, the banker collects a positive compensation once $W_t > W^*$. When $W_t$ is in between $\tilde{W}$ and $W^*$, no transfer occurs between the DIC and the banker. Finally, if $W_t$ falls into the region $[\tilde{W}, \tilde{W}]$, the banker must inject capital to the bank. Over this last interval, the equation (15) becomes

$$rF(W_t) = F'(W_t)\rho W_t + \frac{1}{2}F''(W_t)\frac{B^2}{\mu^2}\sigma^2 + \mu + \max_{c_t \in [-K,0]} \left[-c_t \left( 1 + (1+\alpha)F'(W_t) \right) \right]$$

(17)
Since $F'(W) > -\frac{1}{1+\alpha}$, the last term of the equation (17) is decreasing in $c_t$ and so, at the optimum, $c_t = -K$; the function $F$ satisfies

$$rF(W_t) = F'(W_t) (\rho W_t + (1 + \alpha) K) + \frac{1}{2} F''(W_t) \frac{B^2}{\mu^2} \sigma^2 + \mu + K$$

**Proposition 3** The optimal allocation is characterized through the continuation utility $W$ of the banker whose the dynamic is governed by the following stochastic differential equation

$$dW = (\rho W - U[c(W)]) dt + \frac{B}{\mu} dZ$$

The DIC’s payoff is determined by the function $F$ - solution to

$$\begin{cases} rF(W) = F'(W) (\rho W + (1 + \alpha) K) + \frac{1}{2} F''(W) \frac{B^2}{\mu^2} \sigma^2 + \mu + K & \text{for } W \in [\bar{W}, \tilde{W}] \\ rF(W) = F'(W) \rho W + \frac{1}{2} F''(W) \frac{B^2}{\mu^2} \sigma^2 + \mu & \text{for } W \in (\tilde{W}, W^*) \\ F(W) = F(W^*) - (W - W^*) & \text{for } W > W^* \end{cases}$$

with boundary conditions: $F(\bar{W}) = 0, F'(\bar{W}) = -\frac{1}{1+\alpha}, F'(W^*) = -1$ and $F''(W^*) = 0$.

Compensations for the banker are paid according to the policy

$$c(W) = \begin{cases} W - W^* & \text{if } W > W^* \\ 0 & \text{if } W \in [W, W^*] \\ -K & \text{if } W \in [\bar{W}, \tilde{W}] \end{cases}$$

The bank will be closed in the first time $W$ reaches $\tilde{W}$

$$\tau = \inf \{ t : W_t = \tilde{W} \}$$

**Proof.** See appendix 2. \( \blacksquare \)

## 4 Implementation of the optimal allocation with recapitalization possibility

In the section 3, we characterized the optimal allocation that induces the banker to exert high effort every time. Now, we show that this optimal allocation is implementable through a menu of three regulatory tools: bank chartering, capital regulation and deposit insurance premium. In our implementation, all decisions are contingent on the level of book-value capital $E_t$. In other words, in our implementation, the book-value of capital plays the role of a record-keeping device, as $W_t$ does in the abstract characterization of optimal allocation. For implementation purpose, we define three thresholds $E^*, \bar{E}, \tilde{E}$ which respectively correspond to $W^*, \bar{W}$ and $\tilde{W}$.

**Bank chartering.** It define the initial amount of capital $E_0$ the banker must contribute to open the bank. $E_0$ is determined by $W_0$. Once the banker obtains the charter to set up the bank, she will collect $D$ units of deposits and invest it in the risky loan portfolio. We
assume that only deposits are invested; the initial capital is kept as cash to meet possible future liquidity needs, such as paying deposit insurance premium. The amount of cash grows at the risk-free rate $r$.

**Deposit insurance premium.** It is characterized by a sequence of payment $P_t$ from the banker to the DIC. Specifically, during the infinitesimal time interval $[t, t + dt)$, the banker has to pay

$$
P_{t}dt = \begin{cases} 
(B - (\rho - r)E_t)dt + dR_t \left(1 - \frac{B}{\mu}\right) & \text{for } E_t > \bar{E} \\
(B - (\rho - r)E_t)dt + dR_t \left(1 - \frac{B}{\mu}\right) - \alpha Kdt & \text{for } E_t < \bar{E}
\end{cases}
$$

**Capital regulation.** It determines the rules regarding the policies of dividend, recapitalization and liquidation. The bank is prevented to distribute dividends as long as the amount of capital is not greater than $E^*$. When the book-value of capital is larger than this threshold, all excess capital is distributed as dividends. The DIC order a plan of recapitalization from the bank if the capital level falls below $\bar{E}$. The bank is placed into the liquidation procedure if its amount of capital is less or equal $\bar{E}$.

The above menu of regulations exhibits following properties:

Concerning deposit insurance premium, it is decreasing with the amount of capital. Hence, our insurance premium is risk-based premium. The risk is measured by the amount of capital at the beginning of each period.

Relatively to the capital regulation, we see that our model-implied regulation and the US PCA have several similarities. Indeed, first, since our approach to design the banking regulation is to implement the ex-ante optimal allocation, all actions of the regulator is specified ex-ante by the law and so, the regulators’ discretion is limited. Second, our model-implied regulation implies that restrictions on bank activities become more stringent the less capitalized banks are. Banks with high level of capital (more than $E^*$) would be subject to minimum prudential supervision, they can distribute dividends to shareholders. If banks’ capital fall below a certain level, banks are still allowed to continue in operation but become subject to more intensive supervision and more frequent monitoring such as prohibiting dividend payments, requiring a plan of recapitalization. As banks’ capital fall still lower, the bank’s authorities resolve banks through liquidation. Note that in our model-implied regulation, banks should be forced into resolution when they still have positive capital.

## 5 Conclusion

In this paper, we take the approach of designing prudential regulation of banks as a mechanism to implement the socially optimal allocation proposed by Shim (2006) to study the optimality of the current US Prompt Corrective Action. In a dynamic setting where the regulator (the DIC) can not observe the effort chosen by the banker and can require the banker to inject capital at each period, we first derive the optimal allocation specifying the payments to the banker and the liquidation policy, using the banker’s expected discounted utility as state variable. Then, we show that this allocation can be implemented by a
combination of capital regulation and risk-based deposit insurance premium. From the implementation results, we observe that the PCA version applied in US have several optimal properties.

A Appendix 1: Proof of proposition 1

Define by $V_t$ the total utility the banker expects to get from the allocation $(\tau, c)$ if she chooses the effort strategy $A^*$ conditionally on the information available at time $t \leq \tau$

$$V_t = E \left[ \left( \int_0^\tau e^{-\rho s} U(c_s) ds + e^{-\rho \tau} \tilde{W} \right) \bigg| \mathcal{F}_t \right]$$

and by $\tilde{V}_t$ the one the banker receives if she follows an effort strategy $A$ up to time $t \leq \tau$ and then, switches to the strategy $A^*$

$$\tilde{V}_t = \int_0^t e^{-\rho s} (U(c_s) + v(A_s)) ds + E \left[ \left( \int_t^\tau e^{-\rho(s-t)} U(c_s) ds + e^{-\rho(\tau-t)} \tilde{W} \right) \bigg| \mathcal{F}_t \right]$$

So,

$$\tilde{V}_t = V_t + \int_0^t e^{-\rho s} v(A_s) ds$$

Similarly to (9), we can represent $V_t$ as $V_t = V_0 + \int_0^t e^{-\rho s} G_s dZ_s$. Hence, the dynamic evolution of $\tilde{V}_t$ under the probability measure $P$ is the following

$$d\tilde{V}_t = e^{-\rho t} v(A_t) dt + e^{-\rho t} G_t dZ_t$$

Since $Z_t$ and $Z_t^A$ are related by the equality $dZ_t = dZ_t^A + \frac{1}{\sigma} (\mu A_t - \mu) dt$, under the probability measure $P^A$, $\tilde{V}_t$ evolves according to

$$d\tilde{V}_t = e^{-\rho t} \left( v(A_t) + \frac{G_t}{\sigma} \mu A_t - \frac{G_t}{\sigma} \mu \right) dt + e^{-\rho t} G_t dZ_t^A$$

**Conclusion 1** If $\tilde{V}_t$ is $P^A -$ submartingale, then the effort strategy $A^*$ is suboptimal for the banker

**Proof.** Indeed, the fact that $\tilde{V}_t$ is $P^A -$ submartingale means for all $s \leq t$,

$$E^A \left( \tilde{V}_t \bigg| \mathcal{F}_s \right) \geq \tilde{V}_s$$

Therefore, for all $t \geq 0$,

$$V_0 = \tilde{V}_0 \leq E^A \left( \tilde{V}_t \right)$$

(18)
Note that $E^A \left( \tilde{V}_t \right)$ represents the total utility the banker expects to get at date 0 if she follows a strategy $A$ until the time $t$ and then, follows the strategy $A^*$. Obviously, (18) implies that the strategy $A^*$ is suboptimal compared to $A$. ■

Conclusion 2 If $\tilde{V}_t$ is $P^A$ - supermartingale, then the effort strategy $A^*$ is at least as good as the strategy $A$ for the banker

Proof. Since $\tilde{V}_t$ is $P^A$ - supermartingale, we have

$$E^A \left( \tilde{V}_t \bigg| \mathcal{F}_s \right) \leq \tilde{V}_s$$

for all $s \leq t$. That implies that\(^3\)

$$E^A \left( \tilde{V}_t \right) \leq \tilde{V}_0 = V_0$$

Equation (19) accounts for the total utility the banker expects to get at date 0 if she always follows the strategy $A$ and so, (19) concludes the proof. ■

From two conclusions above, we obtain the necessary and sufficient condition for the optimality of the strategy $A^*$. That is, the drift coefficient of $\tilde{V}_t$ under $P^A$ is non positive:

$$v(A_t) + \frac{G_t}{\sigma} \mu A_t - \frac{G_t}{\sigma} \mu \leq 0 \text{ for all } A_t \in \{0, 1\}$$

It is equivalent to $G_t \geq \sigma \frac{B_t}{P}$. Q.E.D

B Appendix 2: Proof of proposition 2 and 3

For the formal proof, we have to establish the following conclusions

1) the allocation characterized in these propositions is incentive compatible
2) It is optimal among the class of incentive compatible allocations

The incentive compatibility of the characterized allocation is directly derived from the specification of the dynamic evolution of banker’s continuation utility. The proof of the optimality proceeds as follows:

B.1 Upper bound of the DIC’s expected payoff

Here, we will prove that the function $F$ - solution, if exists, to the HJB equation with boundary condition $F(\bar{W}) = 0$ constitutes an upper bound for the expected payoff the DIC can earn from any incentive compatible allocation that delivers the banker an initial expected discounted utility $W_0$.

\(^3\) This conclusion is due to the following result:

Let $M$ be a martingale (submartingale, supermartingale). Then for any elementary stopping time $S \leq T$, we have nearly

$$E \left[ |M_T - M_S| \bigg| \mathcal{F}_S \right] = 0 (\geq 0, \leq 0)$$
Consider any incentive compatible allocation \((\tau, c)\), the expected payoff of the DIC is evaluated by

\[
E \left[ \int_0^\tau e^{-rt} dR_t - \int_0^\tau e^{-rt} c_t dt \right] = E \left[ \int_0^\tau e^{-rt} (\mu - c_t) dt \right]
\]

Define a stochastic process \(M = \{M_t\}\) by

\[
M_t = \int_0^t e^{-rs} dR_s - \int_0^t e^{-rs} c_s ds + e^{-rt} F(W_t)
\]  

(20)

where \(W_t\) is defined by (11) with \(G_t \geq \frac{B}{\mu} \sigma\). We have

\[
dM_t = e^{-rt} (dR_t - c_t dt - rF(W_t)dt + dF(W_t))
\]

Using the dynamic of \(F(W_t)\) given by the equation (13), we get

\[
e^{rt} dM_t = \left( \mu - c_t + F'(W_t) (\rho W_t - U(c_t)) + \frac{1}{2} F''(W_t) G_t^2 - rF(W_t) \right) dt
\]

\[
+ \left( \sigma + F'(W_t) G_t \right) dZ_t
\]

Since \(F(W_t)\) is solution to (14), then \(N_t \leq 0 \ \forall t\), which implies that \(M = \{M_t\}\) is super-martingale. Therefore,

\[
E \left[ \int_0^\tau e^{-rt} (\mu - c_t) dt \right] = E [M_\tau] \leq M_0 = F(W_0)
\]  

(21)

In (21), the first equality stems from \(F(W_\tau) = F(\hat{W}) = 0\) and for the inequality, see footnote (3). Q.E.D.

**B.2 DIC’s expected payoff from the optimal allocation**

Now, we show that the allocation characterized in these propositions provide the DIC with expected payoff exactly equal to \(F(W_0)\).

Notice that if \(M = \{M_t\}\) is martingale, then

\[
E \left[ \int_0^\tau e^{-rt} (\mu - c_t) dt \right] = E [M_\tau] = M_0 = F(W_0)
\]

Therefore, we should prove that the process \(M = \{M_t\}\) defined by (20), if evaluated basing on the optimal allocation, is a martingale. In other words, we have to show that the function \(F\) defined in these propositions is concave function. Indeed:

First, we show that the function \(F\) characterized in the proposition 2 is concave. Define
a function $V(W)$ by $V(W) = F(W) + W$. $V(W)$ satisfies the following differential equation

$$rV(W) = \mu - (\rho - r) W + V'(W) \rho W + \frac{1}{2} V''(W) \frac{B^2}{\mu^2} \sigma^2$$

(22)

for all $W \in \left[\tilde{W}, W^*\right]$. Differentiate this equation to obtain

$$\frac{1}{2} V''(W) \frac{B^2}{\mu^2} \sigma^2 = (\rho - r) \left(1 - V'(W)\right) - V''(W) \rho W$$

Hence, $V''(W^*) > 0$, that implies that in the neighbourhood $(W^* - \varepsilon, W^*)$ of $W^*$, $V''(W) < 0$ and $V'(W) > 0$. We will prove that $V'(W) > 0$ for all $W \in \left[\tilde{W}, W^* - \varepsilon\right]$. Suppose that $V'(W) \leq 0$ for some $W < W^* - \varepsilon$. Let $\hat{W} = \sup\left\{W < W^* - \varepsilon : V'(W) \leq 0\right\}$.

So, over the interval $(\tilde{W}, W^*)$, $V'(W) > 0$ and $rV(W) < rV(W^*) = \mu - (\rho - r) W^* < \mu - (\rho - r) W$. By (22), over this interval $V''(W) < 0$. Thus, $V'(\hat{W}) = -\int_\tilde{W}^{\hat{W}} V''(W)dW > 0$, contradiction. Hence, $V'(W) > 0$ for all $W \in \left[\tilde{W}, W^*\right]$. By (22), for all $W \in \left[\tilde{W}, W^*\right]$, $\frac{1}{2} V''(W) \frac{B^2}{\mu^2} \sigma^2 \leq rV(W) - \mu + (\rho - r) W < rV(W^*) - \mu + (\rho - r) W^* = 0$.

The proof for the proposition 3 is similar. Q.E.D

References


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