Abstract
A series of recent, seminal papers have greatly advanced our understanding of bubbles by modeling greater fool’s bubbles, where rational but asymmetrically informed investors buy overvalued assets hoping to sell at a profit before the crash. The authors of these papers note that some assumptions made to keep prices from revealing private information may be controversial. Prices are either assumed to be unresponsive to sales, which is difficult to justify, or prices must satisfy certain parameter restrictions exactly, which makes bubbles fragile. To avoid these critiques, I add multidimensional uncertainty, so that price movements due to investors’ sales can be mistaken for random day-to-day fluctuations. Temporary confusion allows some investors to sell before the crash, allowing bubbles of arbitrary duration to arise. Thus, my paper supports and advances previous contributions by introducing noisy prices in a tractable way and generating bubbles that are robust to the aforementioned critiques.

JEL Classification Codes: G12, G14
Keywords: Bubbles, Efficient Markets, Coordination, Market Timing

1This paper has benefited enormously from comments by John Conlon, Thomas Jeitschko, Andreas Park, and Juan Rubio-Ramírez. I am also indebted to Leonardo Auernheimer, Marco Bassetto, David Benjamin, Dror Goldberg, Patrick Kehoe, Timothy Kehoe, and participants in seminars at the University of Minnesota, Texas A&M University, the Federal Reserve Bank of Atlanta, the 2006 Midwest Theory Meetings, the 2007 Meetings of the Canadian Economics Association, and the 2007 North American Summer Meetings of the Econometric Society for valuable suggestions. All remaining errors are my own.
1 Introduction

The perception that bubbles can inflate asset prices beyond their fundamental values appears to be widespread among financial market participants and policymakers alike. For instance, central bankers Greenspan (1996), Bernanke (2002) and Trichet (2005) have given speeches about monetary policy in the presence of bubbles, and famed investor Warren Buffett (2001, 2005) has, in different contexts, expressed the view that prices were “out of line” with fundamentals. On the other hand, the idea of bubbles has proven very difficult to reconcile with rigorous economic theory, and this has led many economists to be skeptical about bubbles, and to favor fundamentals-based asset pricing.

Standard asset pricing theory is based on the efficient markets hypothesis (Fama (1965)), by which prices reflect all public information about fundamentals, thus ruling out bubbles. Proponents of fundamentals-based asset pricing point out that even in episodes widely cited as bubbles, like the Dutch Tulipmania (1634-1637), the Mississippi Bubble (1719-1720) and the South Sea Bubble (1720), fundamentals-based interpretations cannot be ruled out (see Garber (2001)). Furthermore, Santos and Woodford (2000) show that, in models with rational agents and symmetric information, bubble-equilibria are fragile and depend on special assumptions. In environments with asymmetric information, no-trade theorems (Milgrom and Stokey (1982)) rule out bubbles in a wide class of environments.

However, a series of recent papers have supported view that bubbles are relevant. For example, Brunnermeier and Nagel (2004) document that, in the late 1990s, hedge funds invested heavily in tech stocks, knowing that they were overvalued. Still, many funds succeeded in timing the market, earning large returns for a while, and selling before the crash. There is also a strand of literature (see Lei, Plott, and Noisure (2001)) documenting that, in experimental settings, bubbles are very pervasive.

Moreover, recent models of bubbles have become increasingly compatible with standard economic theory. A particularly influential line of research includes Abreu and

---

2 In 1996, Greenspan gave his famous irrational exuberance speech. In their 2005 and 2002 speeches, Bernanke and Trichet expressed the view that prices may deviate from fundamentals. It is also well known that Mr. Buffett shunned tech stocks in the 1990s, was criticized during the boom, but vindicated during the bust. Mr. Buffett also expressed the view, in 2005, that several housing markets in the US were overvalued.

3 While the focus of this paper is on theory, on an empirical level, there is also an unresolved debate regarding the validity of econometric methods that have been used to support or refute evidence of bubbles in different datasets (see, among others, Flood and Garber (1980), LeRoy and Porter (1981), Kleidon (1986), Diba and Grossman (1988), West (1987), and Evans (1991)).
Brunnermeier (2003) (AB henceforth), Allen, et al. (1993), and Conlon (2004). In these models, asymmetric information deactivates the backward induction mechanism that typically precludes bubbles in other environments. The key idea is that of a “greater fool’s bubble”, by which it is rational to be a fool and invest in an overvalued asset, as long as there is a good chance of finding a greater fool who will pay even more later. Investors chase profits understanding that they may end up being the greater fool, unable to sell before the crash. In AB, rational agents hold a rapidly appreciating asset, and at some point, observe a private signal revealing that it is overvalued. Importantly, they do not know when others observe the signal. In equilibrium, some sell before the crash and make profits, and others suffer losses. Still, the probability of being in the former group and the growth rate of the bubble are high enough to entice agents to take their chances and knowingly hold an overpriced asset. While details differ in Allen et al. (1993) and Conlon (2004), the core ideas are similar. Asymmetrically informed, rational agents know there is a bubble, but want to ride it, because expected profit is positive.

But the authors of these seminal papers note that the assumptions they make to prevent prices from fully revealing private information may be controversial. Allen et al. (1993) and Conlon (2004) assume that parameters, such as the probabilities of different states of the world and dividends at those states, satisfy exact proportions. A small change in one parameter, holding others constant, makes bubbles collapse. In AB, bubbles are robust to small changes in parameters, but prices are, to some extent, independent of selling.

---

4 Other approaches to modeling bubbles include, among others, bubbles that may last indefinitely in expected value (Blanchard and Watson (1982)), overconfidence (Scheinkman and Xiong (2003)), noise agent risk (Delong, et al (1990)), and asymmetric information with call-option-type compensation for fund managers (Allen and Gorton (1993)). For a survey, see Brunnermeier (2001).

5 In reality, fund managers who stay out of bubbles post lower returns than their peers and, unless they have a Buffett-like reputation, lose customers and/or face pressure to resign. Managers who actively bet against bubbles can have disastrous results, as illustrated by the bloomberg.com story on fund manager Michael Berger, who bought put options on dotcom stocks during the second half of the 1990s. As prices kept rising, quarter after quarter, his options expired worthless. In 1996, faced with horrific losses, he started falsifying performance reports, hoping that the crash would come soon. In January of 2000, only a few months before the crash, he could no longer hide his fraud and went out of business. To avoid jail, he fled to Austria in 2002, where he was finally arrested in 2007.

6 To see in what sense these models are fragile, consider the simple example in Conlon (2004). Two agents sell an asset to each other back and forth. From one period to the next the price of the asset either doubles or falls to zero. Each agent’s beliefs are constructed in such a way that, whenever an agent is buying, his subjective probability of the price doubling (conditional on his information) is 50%, so that she is indifferent between buying, selling, or doing nothing. A small change to the price growth rate or to the probabilities destroys the equilibrium.
pressure. Even though those who sell before the crash do so gradually over an interval of
time, the price is insensitive to these sales, and the bubble continues to grow as though
nobody was selling. The bubble bursts only when all the “lucky” agents have sold.\(^7\) As AB
note, this “invisibility-of-sales” assumption is necessary to generate bubbles. Without it,
prices would reveal all private information as soon as sales began, effectively reducing the
measure of agents who can sell before the crash to zero. Given this, agents would sell as
soon as they observed the signal.

AB, Allen et al. (1993) and Conlon (2004) also coincide in suggesting that if their
models were extended to allow for multidimensional uncertainty, these assumptions could be
avoided and the main results would still hold. In this paper, I follow this hint by constructing
a discrete-time model that is heavily based on AB, but introduces two new elements.\(^8\) First,
prices always reflect selling pressure, and second, prices have an unobservable noisy
component. As in AB, the growth of the bubble is fueled by demand from behavioral agents.
But I assume that this demand, and thus the growth rate of the price, has some randomness.
Also, I assume that as rational agents start selling, the absorption capacity of behavioral
agents is progressively satisfied, and thus, the expected price growth rate falls. Hence, prices
are not fully revealing since slow price growth may mean that rational agents are selling, or
that the realization of the random shock is low.\(^9\)

In my characterization of equilibria, I first examine the case without noise, in which
as soon as one type sells (a type includes those who observed the overvaluation signal in the
same period), all uncertainty is revealed, triggering a crash in the next period. I derive a
parameter restriction under which, without noise, no bubbles arise because agents sell as
soon as they observe the signal. Maintaining this restriction, I increase the amount of noise
so that it may conceal sales of one type, but not more. Then, price growth can be high,

\(^7\) In AB there are also instances where, in equilibrium, a positive mass of lucky agents sell simultaneously in
one instant, and the remaining lucky agents sell gradually over an interval of time. Even in these cases, sales
are not reflected by the price until the mass of shares sold reaches an exogenous threshold.
\(^8\) The discrete-time assumption, in addition to increasing tractability, might be of independent interest, since it
implies that agents have a finite number of trading opportunities before the time of the crash. In some models
(e.g. Allen and Gorton (1993)) continuous time is necessary, precisely because it generates infinitely many
trading opportunities.
\(^9\) Adding noise to keep prices from being fully revealing is common in other areas of financial economics (see,
for example, the papers on herding by Avery and Zemsky (1998) and Park and Sabourian (2006)). The
assumption is also realistic, since many factors besides private information, for example liquidity needs,
contribute to the volatility of trading volume and asset prices. The presence of noise captures the idea that it is
impossible to keep track of the aggregate of all of those factors with accuracy.
revealing that nobody has sold, medium, revealing that maybe one type has sold and maybe not, or low, revealing that at least one type has sold. I show that, even with this minimal amount of noise, there is a region of the parameter space where bubbles of arbitrary duration arise. In this class of equilibria, agents wait the longest before selling if price growth is high, but sell immediately if it is low. If price growth is medium, they also wait for a while, but not as long as with high growth. Agents find this optimal because at all times, those who are not supposed to sell think that their type may have been first to observe the signal, in which case, selling preemptively would mean foregoing large gains. These first equilibria are built under the simplifying assumption that agents cannot reenter the market after selling. Later, I drop this requirement, and analyze how results change if agents can buy and sell at will. This changes matters, because, with forbidden reentry, an agent who sells preemptively misses out on large gains if her type turns out to be first. But with allowed reentry, if she sees that her type was first, she can reenter the market and forego only a fraction of those gains. Hence, the opportunity-cost argument that deters preemptive sales is weakened, and some of the first equilibria must be discarded because they are not “reentry-proof”. Nevertheless, the reentry option does not change results qualitatively, since the opportunity cost associated with preemptive selling can still be large enough to generate arbitrarily long bubbles. Finally, I let noise hide sales by \( z \geq 1 \) types. In this case, strategies that condition actions only on the most recent price ratio are no longer optimal. This is partly because, with minimal noise, if an agent knows that her type is selling, this greatly increases chances that price growth will be low and the crash will happen next period. But as noise increases, sales by one type have a diminishing marginal impact on the probability that the bubble will burst, and thus, it becomes increasingly important for agents to infer how many other types could be selling. Thus, I construct a class of equilibria where agents sell if it has been at least a given number of periods since observing the signal and if the price history implies that up to \( z \) types could be out of the market by the end of the period. Strategies still satisfy a Markov property, since a single variable summarizes what the price history reveals about how many types could be out of the market. I numerically compute the payoffs a player obtains by following this strategy vs. deviating from it, and show that for some parameters this strategy is indeed optimal. Not surprisingly, as noise increases, the rate at which the price needs to grow in order to generate bubbles falls.
In sum, this paper takes models of bubbles one step further by providing microfoundations for assumptions that were previously exogenous, and by validating the conjecture that a model with noisy prices would generate robust bubbles even if prices responded to selling pressure at all times. Indeed, results largely confirm previous findings, since asymmetric information allows bubbles of arbitrary length to arise. The bubbles I present are less deterministic, in the sense that in all equilibria there is a range of possible bursting times, and the realizations of noise determine where in that range the bubble bursts, and how many types manage to sell before the crash. As in AB, in my model price movements may act as coordination devices that can be ignored or trigger sales, and thus, there are multiple equilibria that can serve as a theory of overreaction to news, and fads and fashions in technical analysis.

As mentioned above, many economists eschew the idea of bubbles precisely because of the theoretical weaknesses of existing models. While AB, Allen et al, Conlon, and others have made great progress towards addressing these concerns, some features of the models are still difficult to defend on theoretical grounds. Taking steps towards demonstrating that bubbles still arise without these features shall increase the acceptability of models of greater fool’s bubbles and increase their application to a series of questions, such as optimal policy in the presence of bubbles, or the emergence of bubbles in markets besides the stock market, such as real estate and foreign exchange.\footnote{There are already some applications of models of bubbles. For example, Minguez-Afonso (2007) applies the AB model to currency crises.}

The paper is organized as follows. In sections 2 and 3, respectively, I describe the environment and define equilibrium. In section 4 I construct bubbles with minimal noise and in section 5 I allow noise to hide sales by more than one type. Section 6 concludes.

2 The Model

Time is discrete and infinite with periods labeled $t = \ldots, -1, 0, 1, \ldots$. There are two assets, a risky asset that trades at the price $p$, and a risk-free asset with gross return $R > 1$. For $t \leq 0$, $p_t$ is given by $R^t$, i.e. up to time zero the risky asset appreciates at the risk-free rate, and $p_0$ is normalized to 1. After time 0 news begin arriving about events that increase the
fundamental value of the risky asset. Price growth accelerates in response to the news, and thus, for \( t \geq 1 \), \( \frac{p_t}{p_{t-1}} \) is a random variable with expected value \( G > R \).

There are two kinds of agents, rational and behavioral. The only role of behavioral agents is to fuel the risky asset’s appreciation. Every period, they become more enthusiastic about prospective returns, and no matter how high the price gets, they continue to demand shares until they have bought up to their absorption capacity, which equals \( \kappa \in (0,1) \) shares. Then, there is a unit mass of rational agents with discount factor for future utility given by \( 1/R \). They are risk neutral, and the positions they can take in the risky asset are limited by short sales constraints. Concretely, their positions range between a maximum of 1 and a minimum of 0. At \( t = 0 \), rational agents are fully invested in the asset, i.e. holdings equal one. The model’s ability to generate bubbles will depend on whether these agents keep holding the asset even after they learn that it is overvalued.

Fast appreciation of the asset is justified by fundamentals for some time. However, behavioral agents keep fueling the growth of \( p_t \) even after it surpasses the level granted by fundamentals. This generates a mispricing that is corrected only when the bubble bursts. The first period in which the asset becomes overvalued is denoted by \( t_0 \), which is a random variable with probability function \( \varphi \) given by

\[
\varphi(t_0) = e^{-\lambda t_0} \left( e^\lambda - 1 \right) \quad \text{for all } t_0 = 1, 2, \ldots
\]

with \( \lambda > 0 \). A crucial ingredient of the model is that when \( t_0 \) is realized, it is not perfectly observable. Instead, every period from \( t_0 \) to \( t_0 + N - 1 \), a mass \( 1/N \) of rational agents observe a signal revealing that the overvaluation has begun. This divides the unit mass of rational agents into \( N \) different types, indexed by \( n \in \{t_0, \ldots, t_0 + N - 1\} \). Rational agents know when they observed their signal, but not when others did. That is, once they get their signal they know \( n \), but not \( t_0 \). Given the signal \( n \), the probability that \( t_0 = j \) is given by

\[
\varphi(j \mid n) = e^{-\lambda j} / \left( e^{-\lambda \max\{1, n-(N-1)\}} + \cdots + e^{-\lambda n} \right), \quad \text{for } \max\{1, n-(N-1)\} \leq j \leq n \text{ and zero otherwise.}
\]

Agents also update beliefs as they observe prices, which reflect a mixture of noise

\[\text{In order to facilitate comparison with the literature, I have kept the model as close as possible to AB. Besides having discrete time, the only new ingredients are noise and price responsiveness to sales.}\]

\[\text{This is a discrete version of the distribution with cdf } \Phi(t_0) = 1 - e^{-\lambda t_0}, \forall t_0 \in [0, \infty) \text{ used by AB. } \varphi \text{ assigns the same probability to } t_0 \text{ that } \Phi \text{ assigns to } \{t_0 - 1, t_0\}, \text{ i.e. } \forall t_0 \geq 1, \varphi(t_0) = \Phi(t_0) - \Phi(t_0 - 1).\]
and sales by others. Thus, $I_{n,t}$—which denotes information available to type-$n$ agents at time $t$—is given by the price history $p^{t-1} = (p_0, p_1, \ldots, p_{t-1})$ before observing the signal, and by the signal and price history $(n, p^{t-1})$ after observing the signal. Also, let $h_{n,t}$ denote asset holdings of type-$n$ agents at the end of period $t$ (and the beginning of $t+1$) and $H_t = (h_{0,t} + \cdots + h_{N,t+1})/N$ denote aggregate holdings across all rational agents. As mentioned above, for all $n$, $h_{n,0} = 1$ and because of short sales constraints, for all $n$ and all $t$, $h_{n,t} \in [0,1]$.

From $t = 1$ until the bubble bursts, $p_t / p_{t-1}$ is given by $G + \alpha(e_t - (1 - H_t))$,\(^{13}\) where $\alpha > 0$ is a parameter capturing the extent to which prices respond to sales. The noisy component of the price $e_t$ is uniformly distributed over $[-\overline{\epsilon}, \overline{\epsilon}]$, and $1 - H_t$ denotes rational agents’ cumulative sales (net of repurchases). Clearly, as sales start, price growth falls in expected value, but because of noise, prices reflect these sales imperfectly. For example, if $1/N < 2\overline{\epsilon} < 2/N$, noise can hide at most sales by one type. Price ratios $p_t / p_{t-1}$ above $G + \alpha(\overline{\epsilon} - 1/N)$ reveal that as of time $t$ nobody has sold, below $G - \alpha\overline{\epsilon}$ that at least one type has sold, and for ratios between these two thresholds, it could be that one type has sold, or none.

The bubble bursts once $1 - H_t \geq \kappa$, i.e. once cumulative sales net of repurchases reach $\kappa$, where $\kappa \in (0,1)$. If the bubble bursts at $T$, the post-crash price $p_T$ is given by $p_{T-1}R(G/R)^{(T-t_0)}$. Note that if $\alpha$ is small—so that from $t_0$ until $T$, $p_t / p_{t-1} \approx G$—this post-crash price approximately equals the price that would be observed if there never was a bubble, in which case prices would be $p_t = p_{t-1}(G + \alpha e_t)$ while $1 \leq t \leq t_0 - 1$ and $p_t = p_{t-1}R$ for $t \geq t_0$. After the crash, the price grows at the rate $R - 1$. In sum, for $t > 0$, the price process is given by

\(^{13}\) Several models could give rise to a formula of this kind. For example, supply could be the sum of a constant (capturing behavioral sales) plus sales by rational agents, and demand could be a downward sloping curve that shifts upward over time, by a factor at first equal to $G$ (in expectation), then lower as rational agents sell. These assumptions would make the model more explicit, but also less tractable and less comparable with AB.
Within-period timing is as follows. At the start of period $t$, having observed previous prices $p_{t-1} = \{..., p_{t-2}, p_{t-1}\}$, and if $t \geq n$ the signal $n$, a type-$n$ agent submits orders to buy/sell, or does nothing. Later in the period, once all agents have submitted their orders and $\varepsilon_i$ has been realized, $p_t$ is determined and orders are executed.14

Having described the environment, a question that naturally arises is why, after emphasizing the need for rigorous microfoundations, the model includes behavioral agents. The reason for this inclusion is that I have attempted to introduce noise and price responsiveness in a parsimonious way, making as few changes as possible in order to facilitate comparison with existing literature. That being said, the presence of behavioral traders is not absolutely necessary in order to generate bubbles, as the model can be modified in such a way that all agents optimize and bubbles still arise. Suppose, for instance, that rational traders get an endowment every period and invest it in the risky asset while they believe that it will keep appreciating. This would fuel price growth. Supply could be modeled as follows. Every period, a randomly chosen fraction of agents would experience a shock (representing unforeseen events that increase liquidity needs, such as medical emergencies) that forced them to sell. Individuals would know whether they themselves had been hit by the shock, but not the total fraction of agents also hit by the shock, and this fraction could vary randomly from period to period. Since sales may be mistaken for the

\[
p_t = \begin{cases} 
  p_{t-1} \left( G + \alpha \left( \varepsilon_t - (1 - H_t) \right) \right) & \text{if } 1 - H_s \leq \kappa \quad \forall s \in \{0,1,\ldots,t\} \\
  p_{t-1} R \left( \frac{G}{R} \right)^{-(t-t_0)} & \text{if } 1 - H_t \geq \kappa \quad \text{and } 1 - H_s \leq \kappa \quad \forall s \in \{0,1,\ldots,t-1\} \\
  p_{t-1} R & \text{if } 1 - H_s \geq \kappa \quad \text{for at least one } s \in \{0,1,\ldots,t-1\}.
\end{cases}
\]

---

14 Agents in the model are thus submitting market orders, since they know that their orders will be executed, but they do not know exactly at what price. See Chapter 3 in Brunnermeier (2001) for a description of microstructure models and types of orders (market orders, limit orders, stop orders, etc). During the period of the crash there are so orders to sell that the execution price is far below the last price that agents saw. In these situations, which practitioners call fast markets, from the time an order is submitted until it is executed, the price may have skyrocketed or plummeted. While I assume that, in the crash period, all orders are executed at the post-crash price, nothing would qualitatively change if instead I followed AB and let some shares be sold at the pre-crash price and others at the post-crash price.
realization of the random shock, this model would give rise to same kind of uncertainty as the model with behavioral agents.

3 Equilibrium

The equilibrium concept is Perfect Bayesian Nash Equilibrium (PBNE), consisting of strategies \( h_{n,t} ( h_{n,t-1} | I_{n,t} ) \), which determine next-period asset holdings as a function of current holdings and information, and beliefs \( \mu ( t_0 | I_{n,t} ) \), which are probability distributions over values of \( t_0 \), conditional on prices and, if \( t \geq n \), the signal.

I first define equilibrium beliefs taking strategies \( h_{n,t} ( \bullet | \bullet ) \) as given. \( \mu ( t_0 | I_{n,t} ) \) assigns positive probability only to values of \( t_0 \) that are consistent with \( I_{n,t} \). The set of such values is the support of \( t_0 \) given \( I_{n,t} \) and denoted by \( \text{supp}( t_0 | I_{n,t} ) \). As agents observe signals and prices, they progressively eliminate values of \( t_0 \). The signal \( n \) reduces the support of \( t_0 \) from the set of positive integers to the set \( \{ \max\{1, n - (N - 1)\}, \ldots, n \} \). And the price history \( p^{t-1} \) rules out values of \( t_0 \) as follows: Given \( p^{t-1} \) and (2), agents know \( \varepsilon_s - (1 - H_s) \) for \( s = 1, \ldots, t - 1 \). Given this, initial holdings \( h_{n,0} = 1 \) for all \( n \), and \( h_{n,t} ( \bullet | \bullet ) \), every value of \( t_0 \) implies realizations of \( \varepsilon_s \) for \( s = 1, \ldots, t - 1 \), and holdings \( h_{m,s} \) for all types \( m \in \{t_0, \ldots, t_0 + N - 1\} \). A value of \( t_0 \) is ruled out by \( p^{t-1} \) if \( |\varepsilon_s| > \overline{\varepsilon} \) for at least one \( s \). The set \( \text{supp}( t_0 | I_{n,t} ) \) collects all the values of \( t_0 \) not ruled out through this process. Since all values of \( \varepsilon_s \) with \( |\varepsilon_s| \leq \overline{\varepsilon} \) are equally likely, by Bayes’ rule, the probabilities \( \mu ( t_0 | I_{n,t} ) \) are given by

\[
\mu ( t_0 | I_{n,t} ) = \frac{\varphi ( t_0 )}{\sum_{t_0 \in \text{supp}( t_0 | I_{n,t} )} \varphi ( t_0 )} \quad \text{for all } t_0 \in \text{supp}( t_0 | I_{n,t} ).
\]  

---

\(^{15}\) Intermediate steps to see whether a value \( j \in \text{supp}( t_0, I_{n,t} ) \) are as follows: Fix \( j \) and given \( h_{m,0} = 1 \) for all \( m \in \{ j, \ldots, j + N - 1 \} \), strategies imply \( h_{m,1} ( 1 | I_{n,t} ) \) for all \( m \), and summing across types yields \( H_j \). Having found \( H_j \), and given \( p_j / p_0 \) and (2), \( \varepsilon_j \) can be inferred. Given \( h_{m,j} \) for all \( m \in \{ j, \ldots, j + N - 1 \} \), the same steps can be repeated to find \( \varepsilon \) and so forth up to period \( t - 1 \),
Next, I define the equilibrium strategy $h_{n,t}(\cdot \cdot)$ taking $\mu$ as given. This strategy maximizes $V(h_{n,t-1} \mid I_{n,t})$, the value of holdings given $I_{n,t}$. Maximization is greatly simplified by risk neutrality. If the expected current price $E_{n,t}p_t$ is at least as large as the expected discounted future value $E_{n,t}V(1 \mid I_{n,t+1})/R$ per share—where $E_{n,t}$ denotes expected value given $I_{n,t}$—then $h_{n,t} = 0$ is optimal, and otherwise, $h_{n,t} = 1$ is optimal. The expected value $E_{n,t}V(1 \mid I_{n,t+1})$ is well defined because we know that the value of holdings after the burst is the post-crash price. This gives us an expected terminal value, from which to iterate backwards. In sum, the equilibrium strategy $h_{n,t}(h_{n,t-1} \mid I_{n,t})$ solves

$$V(h_{n,t-1} \mid I_{n,t}) = \max_{h_{n,t}} E_{n,t}p_t(h_{n,t-1} - h_{n,t}) + \frac{1}{R} E_{n,t}V(h_{n,t} \mid I_{n,t+1})$$

subject to (2) and $0 \leq h_{n,t} \leq 1$.

4 Bubble Equilibria with Minimal Noise

In the presence of noisy prices, multiple equilibria are unavoidable, since a given price pattern can either trigger sales, or be ignored, and both of those responses are optimal for an individual agent, as long as others respond to that price pattern in the same way. For this reason, I will not attempt to characterize all the model’s equilibria, and will instead focus on a series of examples that illustrate how bubbles arise.

I will begin, in subsection 4.1, by studying the case without noise, i.e. the case where prices reveal sales as soon as one type exits the market. Restricting attention to a particular class of strategies, I derive a parameter condition, namely that $G/R < \Gamma$, that ensures that agents sell as soon as they observe the signal. In subsection 4.2, maintaining $G/R < \Gamma$, so that there are no bubbles without noise, I raise $\bar{\epsilon}$ so that noise

---

16 As will be shown later, I only consider strategies where, for any price history, agents sell a finite number of periods after observing the signal. This implies that there is a well-defined maximum-possible duration of the bubble. Thus, it is always possible to iterate backwards from that finite period in order to calculate $V$. We could also assume, as AB do, that there is an exogenous maximum duration of the bubble.

17 The fact that, even without noise, if $G/R > \Gamma$, agents do not sell immediately after getting the signal does not contradict AB. This is because, in discrete time, if all agents of the same type sell simultaneously, a positive measure $1/N$ of agents can sell before the crash even if there is no noise. However, as will become clear later, as $1/N$ shrinks, $\Gamma$ does not, implying that as $1/N$ approaches zero, the bubble would have to grow infinitely fast for agents not to sell immediately after the signal.
may conceal sales of one type but not more. Then, price ratios can be high, revealing that nobody has sold, medium, revealing that maybe one type has sold and maybe not, or low, revealing that at least one type has sold. I show that this minimal amount of noise is enough for arbitrarily long bubbles to arise. While this result, at first, relies on the assumption that agents cannot reenter the market after selling, in subsection 4.3, I allow for reentry and show that, although some equilibria vanish, it is still possible to generate arbitrarily long bubbles that are “reentry-proof”.

Before presenting the examples, I make two more assumptions about parameter values that will greatly simplify algebra. First, I assume that the sensitivity of prices to selling pressure $\alpha$ is strictly positive, so that prices reveal information, but small, so that from $t = 1$ until the crash, $p_t / p_{t-1} \approx G$. Second, I let $\lambda \approx 0$, which implies that $\mu(t_0 | I_{n,t})$ approximately equals the inverse of the number of elements in $\text{supp}(t_0 | I_{n,t})$.

4.1 The Case without Noise

Assume that $2\bar{\varepsilon} < 1/N$, which implies that sales must always be detected, since $G - \alpha\bar{\varepsilon}$, the lowest possible price ratio when nobody has sold, is above $G + \alpha(\bar{\varepsilon} - 1/N)$, the highest possible ratio if one type has sold. Also, assume that $1/N < \kappa$, so that sales of the first type do not burst the bubble directly. Consider the following profile of strategies.

**Example 1** For all $n \in \{t_0, \ldots, t_0 + N - 1\}$ the strategy of a type-$n$ agent is:

$$h_{n,t} \left( h_{n,t-1}, I_{n,t} \right) = \begin{cases} 1 & \text{for all } t \in \{1, 2, \ldots, t^* - 1\} \\ 0 & \text{for all } t \geq t^*, \end{cases}$$

where $t^* = \min\{n + \tau^*, \hat{\tau}\}$ and $\hat{\tau} = \min\{t | t \geq 2 \text{ and } (p_{t-1} / p_{t-2}) < G - \alpha\bar{\varepsilon}\}$.

In words, (5) dictates that agents hold the maximum long position until prices reveal that sales have started or $\tau^*$ periods have passed since observing the signal. As soon as one of these two conditions is met, they should sell everything and never re-enter the market. If everybody follows this, type-$t_0$ agents sell at $t_0 + \tau^*$, the price ratio $p_{t_0 + \tau^*} / p_{t_0 + \tau^* - 1}$ reveals these sales, and other types sell at $t_0 + \tau^* + 1$, bursting the bubble.

18 These assumptions greatly increase tractability and in many instances make it possible to characterize equilibria analytically. Furthermore, they affect results quantitatively, but not qualitatively, since payoffs vary continuously with $\alpha$ and $\lambda$. Formulas for general $\alpha$ and $\lambda$ are available upon request.
In Lemma 1, I derive conditions under which all agents find it optimal to follow (5), and derive a threshold $\Gamma$ such that, if $G/R < \Gamma$, the only equilibrium is one where agents sell immediately after getting the signal. Equilibria with $\tau^* \geq 1$ and $G/R < \Gamma$ are ruled out because agents refuse to wait for $\tau^*$ periods after observing the signal, given that price growth under this threshold fails to compensate for the potential loss in the event of a crash.

**Lemma 1** Consider an environment where $\lambda \approx 0$, $\alpha \approx 0$, and $2\bar{\varepsilon} < 1/N < \kappa$. If $G/R < \Gamma$, with $\Gamma = (1 + \sqrt{5})/2$, the strategy profile (5) is an equilibrium if and only if $\tau^* = 0$.

**Proof** Verifying that (5) is an equilibrium has two parts. First, verifying that everybody finds it optimal to sell when (5) dictates that they should sell. Second, verifying that everybody prefers not to sell to when (5) dictates that they should not sell.

For the first part, consider, for any $\tau^* \geq 0$, the decision problem of an agent of arbitrary type $n$ who, at time $n + \tau^*$, sees that there were no sales in previous periods. At this point, she knows that her type must be $n = t_0$. She also knows that other agents of her type are selling, that $1/\alpha + \tau^*$ will reveal those sales, and that the crash will happen at time $n + \tau^* + 1$. Obviously, her expected payoff from selling now $p_{n+\tau^*}$ exceeds the post-crash price $p_{n+\tau^*}(G/R)^{(\tau^*+1)}$ that she will get if she waits. Strategy (5) also dictates that agents should sell once they see that sales have started, which is (weakly) optimal since at that point agents will get the post-crash price no matter what they do.\(^{19}\)

For the second part, first consider a type-$n$ agent at time $t < n$, i.e. before observing the signal. If prices $p_{t-1}/p_{t-2} > G - \alpha\bar{\varepsilon}$, $\text{supp}(t_0 \mid I_{n,t})$ is $\{t - \tau^*, t - \tau^* + 1, t - \tau^* + 2, \ldots\}$. Selling now yields $p_t$, whereas if she waits, the payoff is uncertain. If $t_0 = t - \tau^*$, there will be a crash at $t + 1$, but if $t_0 > t - \tau^*$, there will be no crash at $t + 1$ and she will earn $G$ for at least one more period. Given that $\Pr[t_0 = t - \tau^* \mid t_0 \geq t - \tau^*] = 1 - e^{-\lambda} \approx 0$, since $\lambda \approx 0$, a crash at $t + 1$ is so unlikely that our type-$n$ agent would rather wait than sell.\(^{20}\)

---

19 Once rational agents see that sales have started, they are indifferent between selling and waiting, since they get the post-crash price in either case. I assume that, in this situation of indifference, agents sell. This is unimportant, however, since strict preference for selling can be easily induced, for example by letting some shares to be sold at the pre-crash price, and the rest at the post-crash price.

20 This reasoning implicitly assumes that $t - \tau^* \geq 1$. If this does not hold, though, agents are even less inclined to sell preemptively, since chances of a crash at $t + 1$ are literally zero.
Note that if $\tau^* = 0$, selling preemptively is the same thing as selling before observing the signal. Thus, for $\tau^* = 0$, all possible cases have been covered and thus, the existence of an equilibrium with $\tau^* = 0$ that does not need any restrictions on $G/R$ has been established.

More conditions have to be verified when $\tau^* \geq 1$. In particular, $G/R$ has to be high enough to keep agents who have observed the signal from selling before $n + \tau^*$. The key tradeoff in that situation is best illustrated by considering a type-$n$ individual at time $t = n + \tau^* - 1$, with $p_{t-1} / p_{t-2} > G - \alpha \bar{\epsilon}$. At this point, prices and (5) imply that supp$(t_0 | I_n) = \{n-1, n\}$, and since $\lambda \approx 0$, both values of $t_0$ are roughly equiprobable. An individual type-$n$ agent understands that, if she sells now, she will get $p_s$ and if she waits she will earn (in expected, discounted value) $p_s (G/R)^{-(\tau^*+1)}$ if $t_0 = n-1$, and $p_s G/R$ if $t_0 = n$. Thus, she will find it optimal to deviate from (5) by selling at $n + \tau^* - 1$ if

$$1 > \frac{1}{2} \left( \frac{G}{R} \right)^{-(\tau^*+1)} + \frac{1}{2} \frac{G}{R}. \quad (6)$$

Note two things here. First, since the crash is bigger for longer bubbles, if (6) holds for $\tau^* = 1$, it holds for any $\tau^* \geq 1$. Second, for any $\tau^*$, (6) captures the situation in which incentives to sell preemptively are strongest. In other words, the time at which agents are most strongly tempted to sell preemptively is at time $n + \tau^* - 1$, just one period before they are supposed to sell. At other times $t = n + \tau^* - s$ (with $s \geq 2$) supp$(t_0 | I_n)$ is given by $\{n-s, \ldots, n\}$ and the probability that of a crash at $t+1$ is given by $\Pr[t_0 = t-s | t-s \leq t_0 \leq t] \approx 1/(s+1)$, clearly below $\frac{1}{2}$. Thus, if (6) holds for $\tau^* = 1$, there is no equilibrium where agents follow (5) unless $\tau^* = 0$. Finally, see appendix A for proof that (6) with $\tau^* = 1$ is equivalent to $G/R < \Gamma$, where $\Gamma = (1+\sqrt{5})/2 \approx 1.618$. ■

4.2 Introducing a Minimal Amount of Noise

Henceforth, I will maintain the restriction that $G/R < \Gamma$, which rules out bubbles without noise. In this subsection, I show that, if the amount of noise $\bar{\epsilon}$ is increased and agents play strategies similar to (5), the model generates arbitrarily long bubbles, even if the amount of noise is minimal in the sense that it can hide sales by at most one type. In particular, I assume $1/N < 2\bar{\epsilon} < 2/N$, so that price ratios $p_i / p_{i-1}$ fall into one of three categories: high
if in \([G + \alpha(\bar{e} - 1/N), G + \alpha\bar{e}]\), medium if in \([G - \alpha\bar{e}, G + \alpha(\bar{e} - 1/N)]\) and low if below \(G - \alpha\bar{e}\). High price ratios reveal with certainty that nobody has sold up to (and including) time \(t\), medium ratios are consistent both with nobody having sold and with one type having sold, and low ratios reveal with certainty that at least one type has sold.

As in Example 1, the strategies I will consider in Example 2 are such that everybody sells if they see low price growth, and everybody waits after high or medium price growth. But now, the waiting times depend on whether price ratios are high or intermediate. If price ratios are high, agents wait for \(\tau^*\) periods from the time they observe the signal until they sell, and for intermediate ratios, they wait for \(\tau^{**}\) periods.

**Example 2** For all \(n \in \{t_0, \ldots, t_0 + N - 1\}\) the strategy of a type-\(n\) agent is:

\[
h_{n,t}(h_{n,t-1}, I_{n,t}) = \begin{cases} 
1 & \text{for all } t \in \{1, 2, \ldots, t^* - 1\} \\
0 & \text{for all } t \geq t^*, 
\end{cases}
\]

(7)

where \(t^* = \min\{t_1, t_2, t_3\}\), \(\tau^* \geq \tau^{**} \geq 0\) and

\[
\begin{align*}
t_1 & = \min\{t | t \geq 2\} \quad \text{and} \quad p_{t-1}/p_{t-2} < G - \alpha\bar{e} \\
t_2 & = \min\{t | t \geq 2, \ t \geq n + \tau^{**}\} \quad \text{and} \quad G - \alpha\bar{e} \leq p_{t-1}/p_{t-2} < G + \alpha(\bar{e} - 1/N) \\
t_3 & = \min\{t | t \geq 2, \ t \geq n + \tau^*\} \quad \text{and} \quad G + \alpha(\bar{e} - 1/N) \leq p_{t-1}/p_{t-2} \leq G + \alpha\bar{e}.
\end{align*}
\]

Following (7), agents hold on to the maximum position until one of the following happens: (i) Prices reveal that sales have begun, (ii) it has been at least \(\tau^{**}\) periods since the signal and, given prices, sales may or may not have begun, or (iii) it has been \(\tau^*\) periods since the signal and prices reveal that sales have not begun. As before, once an agent sells, she stays out of the market forever. Figure 1 depicts the type of bubbles generated by these strategies, with \(\tau^* > \tau^{**}\).\(^{21}\) The duration of the bubble is not deterministic, since depending on the realizations of \(\varepsilon_t\), sales may start at any time between \(t_0 + \tau^{**}\) and \(t_0 + \tau^*\). Since \(\tau^* - \tau^{**}\), the width of this window, will be an important variable in the coming analysis, I shall now define \(d \equiv \tau^* - \tau^{**}\).

\[\text{[Insert figure 1 here]}\]

\(^{21}\) \(\tau^* \geq \tau^{**}\) captures the idea that agents are at least as inclined to sell when they think that sales may have started as when they are sure that they have not. It is actually impossible to build an equilibrium with strategies as in (7) and \(\tau^* < \tau^{**}\). Proof of this claim is available upon request.
These strategies have a simple Markov structure, since behavior depends only on how long it has been since observing the signal, and on whether the most recent price ratio is high, medium, or low. Earlier price ratios do affect beliefs, and I take this into account when verifying that strategies are optimal. But fortunately, for appropriate parameter values, agents indeed find it best to follow (7) at all times in the life of the bubble, for any price history. To show this, it is useful to classify the situations in which agents may find themselves into four categories, and to prove one lemma for each category. Thus, in Lemma 2, I verify that type-\(n\) agents find it optimal to sell at \(t = n + \tau^*\) if \(p_{t-1}/p_{t-2}\) is high. In Lemma 3, I derive conditions under which type-\(n\) agents agree not to sell preemptively at \(t < n + \tau^*\) if \(p_{t-1}/p_{t-2}\) is high. Lemma 4 verifies that type-\(n\) agents want to sell at \(t = n + \tau^{**}\) if \(p_{t-1}/p_{t-2}\) is medium, and Lemma 5 derives conditions ensuring that type-\(n\) agents do not want to sell preemptively at time \(t < n + \tau^{**}\) if \(p_{t-1}/p_{t-2}\) is medium.

**Lemma 2** For all \(G/R \in (1, \Gamma)\) and \(\tau^* \geq 1\), type-\(n\) agents find it optimal to sell at \(t = n + \tau^*\) if \(p_{t-1}/p_{t-2}\) is high.

**Proof** Only agents of type \(n = t_0\) may find themselves still in the market \(\tau^*\) periods after observing the signal. At this point, they know that their type is \(n = t_0\) since \(p_{t-1}/p_{t-2}\) could not possibly be high if others had observed the signal before period \(n\). At time \(t = t_0 + \tau^*\), knowing that other agents of type \(t_0\) are selling, a type-\(t_0\) individual faces the following sell-or-wait trade-off. If she sells at \(t\) her payoff will be \(p_t\) and if she waits, with probability \(\pi \equiv (1/N)/(2\bar{\pi})\) the price ratio \(p_t/p_{t-1}\) will be low, precipitating a crash at \(t + 1\) (and resulting in an expected discounted payoff of \(p_t(G/R)^{-(\tau^{**+1})}\)), and with probability \(1 - \pi\), \(p_t/p_{t-1}\) will be medium, in which case \(d + 1\) types will sell at time \(t + 1\). If \(d + 2 < \kappa N\), sales triggered by the medium price will not burst the bubble and it will be possible to sell at the higher expected discounted price \(p_t G/R\). Thus, if \(d + 2 < \kappa N\), selling is preferable to waiting if

\[
1 > \pi \left( \frac{G}{R} \right)^{-(\tau^{**+1})} + (1 - \pi) \frac{G}{R}.
\]
Since $\pi \geq 1/2$ this condition is less stringent than (6), and thus holds for all $G/R \in (1, \Gamma)$ and $\tau^* \geq 1$. (For $\tau^* = 0$, (8) holds for all $G/R \in (1, \Gamma)$ if $\pi \geq \Gamma - 1$.) If $d + 2 \geq \kappa N$, medium price ratios trigger enough sales to burst the bubble, and thus (8) becomes $1 > (G/R)^{-\left(\tau^* + 1\right)}$, which of course holds for any $\tau^* \geq 0$.

Lemma 2 shows that, following fast price growth, agents want to sell when the strategy dictates that they should sell. This result falls out easily and requires no restrictions beyond $G/R < \Gamma$. In contrast, keeping agents who should not sell from selling is not as easy. Preemptive sales may be tempting because, while one waits, there is often a good chance that an earlier type is selling and will precipitate a crash. In Lemma 3, I show that, despite this risk, agents are willing to wait if two conditions hold. The first condition says, roughly, that the bubble must grow quickly and that it is must be likely that, when one type sells, the price ratio will be intermediate rather than low. The second condition is that $d/N$ should be small relative to $\kappa$, so that sales following medium price growth do not burst the bubble.

**Lemma 3** If $1 < (1 - \pi/2)G/R + \pi/2(G/R)^{-\left(\tau^* + 1\right)}$, $d + 2 < \kappa N$, $t < n + \tau^*$ and $p_{t-1}/p_{t-2}$ is high, type-$n$ agents find it optimal not to sell at time $t$.

**Proof** See appendix B.

While the proof is in appendix B, here, I sketch the main ideas. First, I argue that preemptive sales are tempting only if all of the $d + 1$ ratios before $t$ are high. From the point of view of type-$n$ agents, only in those cases, there is a positive probability $(j + 1)^{-1}$ that $t_0$ equals $n - j$. In other cases, the price history rules out $t_0 = n - j$, and thus, type-$n$ agents know that nobody is selling at $t$, and that a crash at $t + 1$ is impossible.

Next, I show that, if type-$n$ agents prefer waiting to selling at time $t = n - j$ (with $p_{t-1}/p_{t-2}$ being high) for $j = 1$, then, they also do for any $j > 1$. To see this, it is useful to first consider the sell-or-wait trade-off of a type-$n$ agent at time $t = n - 1$ (with high $p_{t-1}/p_{t-2}$). In this case, $\text{supp}(t_0 | I_n, t) = \{n-1, n\}$. If the agent waits, with probability $\pi/2$ the bubble will burst at $t + 1$ (this is the scenario where $t_0 = n - 1$ and sales of type-$t_0$ agents

---

22 Again, in the special case $t_0 = 1$, type-1 agents know their type since period 1.
push $p_t / p_{t-1}$ below $G - \alpha c$, while with probability $1 - \pi / 2$ they will sell at $t+1$ and profit from one more period of appreciation. Thus, for $j = 1$, waiting is optimal if

$$1 < \frac{1}{2} \left( \pi \left( \frac{G}{R} \right)^{-(\tau + 1)} + (1 - \pi) \frac{G}{R} \right) + \frac{1}{2} \frac{G}{R}. \quad (9)$$

From here, note that if $j > 1$ the probability of a crash at $t+1$ falls to $\pi / (j+1)$, and their preference for waiting rather than selling preemptively is even stronger if $j > 1$.

Finally, note that (9) only applies to cases with $d + 2 < \kappa N$ since, when $t_0 = n-1$ and $p_t / p_{t-1}$ is medium, sales of type $t_0$ plus the $d + 1$ types that sell at $t+1$ do not burst the bubble. If $d + 2 \geq \kappa N$, incentives to sell preemptively are too strong, and agents refuse to wait $\tau^* > 0$ periods. (See the last paragraph of the proof in appendix B for details on this.)

In sum, Lemma 3 precludes preemptive sales if the last price ratio is high, (9) holds and $(d + 2) / N$ is less than $\kappa$. For large $\tau^*$, (9) approximates $1 < G / R(1 - \pi / 2)$, which is satisfied by many pairs $(G / R, \pi)$ with $1 < G / R < \Gamma$ and $\frac{1}{2} < \pi < 1$. The fact that $d + 2$ must be less than $\kappa N$ indicates that long-lived bubbles, i.e. equilibria with a large $\tau^*$, may arise only if $\tau^{**}$ is also large. This suggests that the key to generating bubbles will be precluding preemptive sales after medium price ratios. But before discussing that, let us quickly prove Lemma 4, by which type-$n$ agents want to sell at time $n + \tau^{**}$ if $p_{n+\tau^{**}-1} / p_{n+\tau^{**}-2}$ is medium.

**Lemma 4** For any $\tau^{**} \geq 0$ and any $G / R < \Gamma$, type-$n$ agents find it optimal to sell at time $t = \min \{2, n + \tau^{**} \}$ if $p_{t-1} / p_{t-2}$ is medium.

**Proof** Consider the wait-or-sell choice of a type-$n$ agent at $t = n + \tau^{**}$. If, in addition to $p_{t-1} / p_{t-2}$ being medium, $p_{t-2} / p_{t-3}$ is also medium, values of $t_0$ other than $n$ and $n-1$ can be ruled out, because if $t_0$ was below $n-1$, at least two types would have sold at $t-1$, making $p_{t-1} / p_{t-2}$ low. Similarly, if $p_{t-2} / p_{t-3}$ is high and $p_{t-3} / p_{t-4}$ is medium, all values of $t_0$ but $n$ and $n-1$ are inconsistent with $p_{t-2} / p_{t-3}$ being high. In these cases, type-$n$ agents see that, if they are second ($t_0 = n-1$) the bubble will burst next period for sure, and if they are first ($t_0 = n$) the bubble will burst with probability $\pi$. Selling is preferable if
1 > \frac{1}{2} \left( \frac{G}{R} \right)^{-\tau^{**} + 2} + \frac{1}{2} \left[ \pi \left( \frac{G}{R} \right)^{-\tau^{**} + 1} + (1 - \pi) \frac{G}{R} \right], \quad (10)

which holds for any \( G / R < \Gamma, \pi \in (0.5, 1) \) and \( \tau^{**} \geq 0 \). While (10) has been derived for price histories implying \( \text{supp}(t_0 | I_{n,t}) = \{n-1, n\} \), incentives to sell are even stronger for other histories. If \( \frac{p_{t-1}}{p_{t-2}} \) is the first medium price ratio after \( k \) consecutive high ratios (with \( k \in \{2, \ldots, d+1\} \)), \( \text{supp}(t_0 | I_{n,t}) \) is given by \( \{n-k, \ldots, n\} \), and thus the probability that the crash will not happen next period falls from \((1-\pi)/2\) to \((1-\pi)/(k+1)\). Inequality (10) does not apply to the special case with \( n = 1 \), since, in that case, type-1 agents know their type with certainty. Still, if \( \tau^{**} \geq 1 \), (10) guarantees that they want to sell at time \( 1 + \tau^{**} \). Finally, note that if \( t > n + \tau^{**} \), type-\( n \) agents know for sure that at least two types (their own type \( n \) and type \( n+1 \)) will sell in the current period, guaranteeing a crash at \( t + 1 \).

Since, intuitively, agents should be more inclined to sell after medium than high prices, and since Lemma 3 already needs parameter restrictions to preclude preemptive sales after high prices, it is not surprising that more restrictions are needed to rule out preemptive sales after medium prices. Indeed Lemma 5 rules out such sales only if \( \pi G / R > 1 \).

**Lemma 5** Suppose that \( t < n + \tau^{**} \) and that \( \frac{p_{t-1}}{p_{t-2}} \) is medium. If \( \pi G / R > 1 \), there exist a threshold \( d \) such that if \( \tau^{**} - \tau^{*} \geq d \), type-\( n \) agents find it optimal not to sell at time \( t \).

**Proof** See appendix B.

Once more, I sketch the main ideas here and provide details in Appendix B. The proof proceeds by finding, among all the possible scenarios with \( t < n + \tau^{**} \) and a medium \( \frac{p_{t-1}}{p_{t-2}} \), the one where type-\( n \) agents are most tempted to sell. Then, I derive conditions under which, even in that worst-case scenario, type-\( n \) agents choose to wait.

---

23 In the exceptional case where \( n = 1 \), type-\( n \) agents know that they were first to observe the signal, and thus (10) does not apply. They will still sell at time \( n + \tau^{**} \) if \( \tau^{**} \geq 1 \) or if \( G / R \) and \( \pi \) are such that (8) holds for \( \tau^{**} = 0 \). If \( G / R \) and \( \pi \) are such that (8) fails for \( \tau^{**} = 0 \), it is still possible to obtain equilibria with \( \tau^{**} = 0 \), as long as type-\( n \) agents sell at time

24 In addition, the expected size of the crash increases, since the equivalent of (10) for \( k \in \{2, 3, \ldots, d+1\} \) is

\[
1 > \frac{1}{2} \left( \frac{G}{R} \right)^{-\tau^{**} + 2} + \frac{1}{2} \left[ \pi \left( \frac{G}{R} \right)^{-\tau^{**} + 1} + (1 - \pi) \frac{G}{R} \right] + \sum_{j=1}^{k-1} \left( \frac{G}{R} \right)^{-\tau^{**} + 2} + \left[ \pi \left( \frac{G}{R} \right)^{-\tau^{**} + 1} + (1 - \pi) \frac{G}{R} \right].
\]
In appendix B, I show that type-\( n \) agents are most tempted to sell preemptively when \( t = n + \tau^* - 1, \tau^* \geq 1 \), \( p_{t-1}/p_{t-2} \) is medium, and the \( d + 1 \) price ratios immediately prior to \( p_{t-1}/p_{t-2} \) are all high. For these price histories, \( \text{supp}(t_0 | I_{n,t}) \) contains the \( d + 3 \) elements \( \{n-(d+2), \ldots, n\} \). In the first \( d + 1 \) cases, i.e. if \( n-(d+2) \leq t_0 \leq n-2 \), agents with signal \( n-2 \) or earlier have been delaying their sales because of high price ratios preceding \( p_{t-1}/p_{t-2} \). Since \( p_{t-1}/p_{t-2} \) is intermediate, they will sell at time \( t \) together with agents of type \( n-1 \), causing a crash at \( t + 1 \). If \( t_0 = n-1 \), only agents of type \( n-1 \) will sell at \( t \), and the bubble will burst at \( t+1 \) with probability \( \pi \), and at \( t+2 \) with probability \( 1 - \pi \). Thus, in all of the \( d + 2 \) cases with \( t_0 < n \), it would be a good idea to sell preemptively. Therefore, for waiting to be optimal, the expected payoff \( W_d \) for the case \( t_0 = n \) must be so large that

\[
1 < \frac{\left( \frac{G}{R} \right)^{-(\tau^*+2)} + \cdots + \left[ \pi \left( \frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \left( \frac{G}{R} \right) \right] + W_d}{d+3}.
\]  

(11)

\( W_d \) depends on \( G/R \) and on how long the bubble keeps growing after \( t \). Potentially, this could be up to \( d+1 \) periods. However, in expected value, the bubble will burst sooner than that, since every period from \( t \) to \( t+d+1 \), there is a probability \( \pi \) that prices will be high and growth will continue, and a probability \( 1 - \pi \) that prices will be intermediate, trigger sales, and cause a crash. Taking this into account, \( W_d \) is given by

\[
W_d = (1-\pi) \frac{G}{R} \left[ 1 + \pi \frac{G}{R} + \left( \pi \frac{G}{R} \right)^2 + \cdots + \left( \pi \frac{G}{R} \right)^d + \left( \pi \frac{G}{R} \right)^{d+1} \right].
\]  

(12)

The crucial element of the proof is that, if \( \pi G/R > 1 \), \( W_d \) grows exponentially as \( d \) increases. This ensures that (11) will hold if \( d \) is big enough. To see this, note that the right-hand-side of (11) is decreasing in \( \tau^* \). Thus, if (11) holds in the limit as \( \tau^* \) approaches infinity, it also holds for lower \( \tau^* \). For large \( \tau^* \), (11) is approximated by

\[
d + 3 < (1-\pi) \left( \frac{G}{R} \right) + W_d.
\]

If \( \pi G/R > 1 \), as \( d \) increases, the left-hand side grows linearly, and the right-hand side exponentially. Hence, there is a threshold value \( \overline{d} \) such (11) holds whenever \( d \geq \overline{d} \).
In sum, Lemmas 2 and 4 require no restrictions beyond $G/R < \Gamma$ in order to ensure that agents want to sell when (7) dictates so. In contrast, by Lemmas 3 and 5, agents will, in some situations, refuse to wait as prescribed by (7), unless $d < \kappa N - 2$, (9), and (11) hold. It remains to see whether all conditions can be simultaneously satisfied, which is not obvious, given that (9) tends to hold when $d$ and $\pi$ are small, while (11) tends to hold in the opposite case. Nonetheless, Proposition 1 establishes that the inequalities are compatible. In fact, for parameters within a given region, bubbles arise for arbitrarily large values of $\tau^*$ and $\tau^{**}$.

**Proposition 1** Suppose that $\pi \in (\frac{1}{2},1)$, $G/R < \Gamma$, $d < \kappa N - 2$, $1 < G/R(1 - \pi/2)$ and $\pi G/R > 1$. If agents’ strategies are given by (7), and $\pi \in (1/\Gamma, (2\Gamma - 2)/\Gamma)$, there exist values of $G/R$ for which bubbles of arbitrary duration arise. Precisely, for parameters in this region, given any integer $w > 0$, there exist equilibria with $\tau^* > \tau^{**} > w$.

**Proof** Given that $G/R < \Gamma$, (8) and (10) hold for all $\tau^* \geq 1$ and $\tau^{**} \geq 0$. Since $1/\pi < G/R$, there exists a $\tilde{d}$ such that (11) obtains for all $d \geq \tilde{d}$. And if $2/(2 - \pi) \leq G/R$, (9) holds for all $\tau^* \geq 0$. Combining these inequalities, we get

$$\max\{1/\pi, 2/(2 - \pi)\} < G/R < \Gamma.$$  \hspace{1cm} (13)

The set of pairs $(G/R, \pi)$ that satisfies (13) is not empty, given that

$$\max\left\{\frac{1}{\pi}, \frac{2}{2 - \pi}\right\} = \begin{cases} \frac{1}{\pi} & \text{if} \quad \frac{1}{2} \leq \pi \leq \frac{2}{3} \\ \frac{2}{2 - \pi} & \text{if} \quad \frac{2}{3} \leq \pi \leq 1 \end{cases}$$

and thus, for $\pi \in (1/\Gamma, (2\Gamma - 2)/\Gamma) \approx (0.618, 0.764)$, $\max\{1/\pi, 2/(2 - \pi)\}$ is below $\Gamma$.

For any pair $(G/R, \pi)$ satisfying (13), bubbles can be constructed as follows. Let $\tilde{d}$ be the first integer for which $d + 3 < (1 - \pi)(G/R) + W_d$. Given $\kappa$, $N$ can always be made so large that $\tilde{d} \leq \kappa N - 2$. Also, $\pi$ and $N$ imply a value of $\epsilon$, namely $\epsilon = (2\kappa N)^{-1}$. Because $2/(2 - \pi) \leq G/R$, $\tau^*$ can grow indefinitely without ever violating (9), and thus, for each $d$ with $\tilde{d} \leq d \leq \kappa N - 2$, there are as many equilibria as integers $k \geq 0$, since all pairs $(\tau^{**}, \tau^*) = (k, k + d)$ satisfy equilibrium requirements.■

Note that, while $\tau^*$ can be arbitrarily large, this does not mean that the equilibria constructed above rely on the possibility that bubbles may last forever. There is always a
well-defined maximum duration of the bubble, which is $\tau^*+2$ periods. An exogenous maximum duration of the bubble $\tau$ could be added to the model, and would have no effect as long as it was greater than $\tau^*+2$.

It should also be noted that, as previously mentioned, multiplicity is unavoidable. For the parameter region described in Proposition 1, there are equilibria with very long-lived bubbles, equilibria with very short-lived bubbles, and anything in between. For other parameter values, it is possible to reduce the extent of multiplicity by lowering the maximum $\tau^*$ that can be supported in equilibrium. For example, if $G/R$ is below $2/(2-\pi)$, (9) only holds for $\tau^* \leq -[\ln(2-(2-\pi)G/R)-\ln \pi]/\ln(G/R)-1$. Unfortunately, one cannot rule out equilibria where agents sell immediately, or very soon after observing the signal. Thus, $(\tau^{**}, \tau^*)=(0,1)$ is an equilibrium whenever $G/R<\Gamma$, and for many combinations of parameter values $(\tau^{**}, \tau^*)=(0,0)$ is also an equilibrium.

To illustrate these sets of equilibria, consider an example with $G/R=1.6$, $N=100$, $\overline{\epsilon}=1/150$ (implying $\pi=3/4$), and $\kappa=1/4$. For these values, $(\tau^{**}, \tau^*)=(k,k+d)$, with $k \geq 0$ and $8 \leq d < 23$ satisfy all the relevant inequalities. Further, $(\tau^{**}, \tau^*)=(0,j)$, for all $j \in \{0,1,\ldots,d-1\}$ are also equilibria. These are parameters for which bubbles can be short, long, or intermediate. A maximum $\tau^*$ exists if $\overline{\epsilon}$ is reduced by a little bit, so that $\pi=0.75+10^{-13}$. Then, (9) only holds for $\tau^* \leq 61$, and no equilibria with $\tau^*$ above 61 exist.

In sum, examples 1 and 2 suggest that noise greatly increases the model’s ability to generate bubbles. With $G/R<\Gamma$, the only equilibrium in example 1 is one where agents sell as soon as they see the signal, whereas in example 2, even though noise can only hide sales by one type, bubbles of unbounded duration arise.\(^{25}\)

4.3 Re-entering the Market after Selling

Thus far, only trigger strategies have been admissible. This has been useful to simplify the analysis, but is unrealistic. Except for markets such real estate, where high transaction costs

\(^{25}\) It is interesting to note that, if noise cannot even hide one sale, i.e. if $1/N > 2\overline{\epsilon}$, it is impossible to build a version of example 2 where agents wait $\tau^*$ if $\epsilon \in (\overline{\epsilon}(1-2\pi), \overline{\epsilon}]$ and $\tau^{**}$ if $\epsilon \in [-\overline{\epsilon}, \overline{\epsilon}(1-2\pi))$, i.e. wait $\tau^*$ periods if prices are in the top $\pi$ fraction of $[-\overline{\epsilon}, \overline{\epsilon}]$ and $\tau^{**}$ for prices in the lower $1-\pi$ fraction of $[-\overline{\epsilon}, \overline{\epsilon}]$. Attempts to generate bubbles in this fashion would fail, because type-$n$ agents’ decisions at time $n+\tau^*-1$, would be governed by (6) instead of (9), and thus, $\tau^*$ could only be zero.
preclude frequent trading, frequent trading is the norm in markets such as stock markets, foreign exchange markets, and others where transaction costs are small.

Moreover, restricting attention to trigger strategies may not be innocuous. For instance, in the situation captured by (11), a type-$n$ agent weighs selling vs. waiting at time $t = n + \tau^{**} - 1$, having observed a medium $p_{t-1}/p_{t-2}$, and $d+1$ consecutive high price ratios $p_{t-2}/p_{t-3}, \ldots, p_{t-(d+2)}/p_{t-(d+3)}$ before. Waiting at $t$ implies getting the post-crash price with probability $(d+1+\pi)/(d+3)$, gaining one more period of appreciation with probability $(1-\pi)/(d+3)$, and gaining $W_d$ with probability $1/(d+3)$. In (11), waiting is optimal because, even though selling is likely to avoid losses, the opportunity cost $W_d$ is potentially huge. Allowing reentry reduces this expected opportunity cost, and may tilt the balance in favor or selling preemptively, since a type-$n$ agent who sold at $t$ and then saw that $t_0 = n$, could reenter the market at $t+1$, hence foregoing only part of $W_d$. Specifically, a type-$n$ agent who sold at $t$ would stay out if $p_t/p_{t-1}$ was low or medium and reenter if $p_t/p_{t-1}$ was high. In this case, which would happen with probability $\pi/(d+3)$, she would earn an expected discounted payoff $W_{d-1}$. Thus, the equivalent of (11) for allowed reentry would be

$$d + 3 - \pi + \pi W_{d-1} < \left(\frac{G}{R}\right)^{-(\tau^{**}+2)} + \cdots + \left[\pi\left(\frac{G}{R}\right)^{-(\tau^{**}+1)} + (1-\pi)\left(\frac{G}{R}\right)\right] + W_d.$$  \hfill (14)

Like (11), (14) holds for all $\tau^{**}$ if it holds in the limit as $\tau^{**}$ approaches infinity, in which case it can be rewritten as

$$d + 3 - \pi < (1-\pi)\left(\frac{G}{R}\right) + W_d - \pi W_{d-1},$$

which, using (12), is equivalent to

$$d + 3 - \pi < (1-\pi)\left(\frac{G}{R}\right) + (\pi G)^d (G-\pi) + (1-\pi)^2 G \frac{(\pi G)^d - 1}{\pi G - 1}.$$  \hfill (15)
As in (11), the left-hand side grows linearly with $d$ and if $\pi G / R > 1$, the right-hand side grows exponentially. Thus, there is a positive integer $\bar{d} > d$ such that (15), and hence (14), obtain for all $d \geq \bar{d}$. However, once reentry is possible, equilibria with $\bar{d} \leq d < \bar{d}$ vanish.\(^{26}\)

Other than the fact that the minimum $\tau^* - \tau^{**}$ is lengthened from $d$ to $\bar{d}$, there are no new requirements that equilibria with long bubbles must satisfy once reentry is allowed. Agents who, following (7), sell at $t = t^*$, never want to reenter at $t + 1$, because, as they (correctly) believe, the bubble will burst no later than period $t + 2$. And, if before reentry was allowed, type-$n$ agents did not sell at $t = n + \tau^*-j$ (for $j \geq 1$) following $d + 1$ consecutive high price ratios $p_{t-1}/p_{t-2}/\ldots/p_{t-(d+1)}/p_{t-(d+2)}$, they still prefer waiting to selling once reentry is allowed. (See appendix C for details.)

In conclusion, reentry makes a quantitative difference, not a qualitative one. The mechanism that protected bubbles from preemptive sales under forbidden reentry still works, and bubbles with arbitrarily large $\tau^*$ still arise. For instance, in the numerical example with $G / R = 1.6$, $N = 100$, $\pi = 3/4$, and $\kappa = 1/4$, we have $\bar{d} = 16$. $(\tau^{**}, \tau^*) = (k, k + d)$, with $k \geq 0$ and $16 \leq d < 23$ satisfy all necessary inequalities, and equilibria from example 2 with large $\tau^{**}$ and $8 \leq d < 16$ are discarded, because they are not “reentry proof”.

5 A More General Specification of Noise

Since bubbles arise even if noise can hide sales from only one type, intuition suggests that this should also be the case with greater amounts of noise. Generalizing the analysis, in addition to being interesting in its own right, makes it possible to relax some of the more stringent parameter restrictions needed to generate bubbles in section 4. Specifically, in previous examples, bubbles are long and reentry-proof only if $G / R > 3/2$. In reality, average appreciation rates above 50 percent per period seem unrealistic unless a period is thought of as being a fairly long time. With more noise, the lower bound on $G / R$ can be reduced to more plausible levels, and periods can thus be interpreted as being shorter.

\(^{26}\) Note that (15) keeps type-$n$ agents from selling at $t = n + \tau^{**}-j$, with medium $p_{t-1}/p_{t-2}$ also in cases with $j > 1$ and/or less than $d + 1$ high price ratios before $p_{t-1}/p_{t-2}$. This is because both of these changes decrease the probability of a crash at $t + 1$, and thus selling preemptively is more likely to imply missing out on gains than avoiding the crash.
Furthermore, when noise can only hide one sale, the model is discrete in a very coarse way, since sales of one type have a big impact on the probability of a crash. As more noise is added, periods become shorter, the single by a single type become decreasingly important, and thus the model becomes somewhat more similar to the continuous-time AB model.

From now on, I let \( 2\varepsilon = (\overline{z} + 1)/N \), so that noise can hide sales of an arbitrary number of types \( \overline{z} \).  Working with an arbitrary \( \overline{z} \) requires that the three categories of price ratios high, medium, and low be expanded to describe the different messages that a price ratio can now communicate. Let us thus define the function \( c : (\alpha, G + \alpha \varepsilon) \to \{1, \ldots, \overline{z} + 1\} \) by splitting \([G - \alpha \varepsilon, G + \alpha \varepsilon]\) into \( \overline{z} + 1 \) subintervals, setting \( c(p_t / p_{t-1}) = z \) if \( p_t / p_{t-1} \in (G + \alpha(\varepsilon - (z + 1)/N), G + \alpha(\varepsilon - z/N)] \), for \( z \in \{0, \ldots, \overline{z}\} \) and \( c(p_t / p_{t-1}) = \overline{z} + 1 \) if \( p_t / p_{t-1} \leq G - \alpha \varepsilon \). The interpretation of these categories is the same as in the minimal-noise case: If \( c(p_t / p_{t-1}) = z \) (for \( z = 0, \ldots, \overline{z} \)), \( p_t / p_{t-1} \) is consistent with anywhere from 0 to \( z \) types being out of the market at the end of \( t \), whereas if \( c(p_t / p_{t-1}) = \overline{z} + 1 \), everybody knows, with probability one, that at least one type has left (and not reentered) the market.

The strategies described by (7) are simple and intuitive and simple, but unfortunately, they cannot be extended to arbitrary amounts of noise in a straightforward manner. Consider, for instance, a version of (7) expanded with \( \overline{z} \) waiting times \( \tau_0^*, \ldots, \tau_{\overline{z}}^* \) satisfying \( \tau_k^* \geq \tau_{k+1}^* \geq 0 \) for \( k \in \{0, \ldots, \overline{z} - 1\} \). If at \( t = t_0 + \tau_0^* \), \( c(p_{t-1} / p_{t-2}) = 0 \), type-\( t_0 \) agents know that their type was first to observe the signal, and that they are supposed to sell. But if \( \overline{z} \) is large, even if other type-\( t_0 \) agents sell at \( t \), \( p_t / p_{t-1} \) is unlikely to trigger a crash at \( t + 1 \), and thus, waiting for one more period before selling might very well be optimal. This contrasts with example 2, where, if a type-\( t_0 \) agent knows that all other type-\( t_0 \) agents are selling at \( t \), it is optimal for her to go along with the sale, since the probability that \( p_t / p_{t-1} \) falls below \( G - \alpha \varepsilon \) is more than \( \frac{1}{2} \). Another example of why the strategies in the minimal-noise case do not admit a straightforward extension, is that it is easy to imagine situations where type-\( n \) agents may not want to sell at time \( t = n + \tau_{\overline{z}}^* \), even if \( c(p_{t-1} / p_{t-2}) = \overline{z} \). For instance, if \( \overline{z} \) is

\[27\] I assume that \( \overline{z} \) is an integer. Removing this assumption complicates formulae without adding insight. Also, it is technically possible that \( \overline{z} + 1 \) types might sell and the price ratio would not fall below, but instead equal \( G - \alpha \varepsilon \). However, I ignore this, because it is a measure zero event.
large, \( c(p_{t-2}/p_{t-3}) = 0 \), and \( c(p_{t-3}/p_{t-4}) = \bar{z} \), type-\( n \) agents would know that at most two types would be selling at time \( t \) and thus, the probability of a crash at \( t+1 \) would be low. These examples suggest that strategies can no longer condition sales on just the most recent price ratio, since agents are only willing to sell at \( t \) if a crash at \( t+1 \) is likely, or in other words, if it is probable that a large number of types (relative to \( \bar{z} \)) will be out of the market by the end of \( t \).

Hence, I will construct a class of equilibria where agents sell at time \( t \) only at least a given number of periods has passed since observing the signal and if, given an selling agent’s information, up to \( \bar{z}+1 \) types could be out of the market by the end of time \( t \).

**Example 3** The equilibrium strategy of a type-\( n \) agent is:

\[
 h_{n,t}(h_{t-1},I_{n,t}) = \begin{cases} 
 1 & \text{for all } t \in \{1,2,\ldots,t^*+1\} \\
 0 & \text{for all } t \geq t^*,
\end{cases}
\]

with \( t^* = \min\{t_1,t_2\} \), and \( t_1 \) and \( t_2 \) given by

\[
 t_1 = \min \{ t \geq 2 \mid c(p_{t-3}/p_{t-2}) = \bar{z}+1 \}
\]

\[
 t_2 = \min \{ t \geq 1 \mid t \geq n+\tau^* \text{ and up to } \bar{z}+1 \text{ types could be out of the market by the end of } t \}.
\]

Agents sell if certain that sales have begun, or if it has been at least \( \tau^* \) periods since the signal and, given the price history and strategy profile, \( \bar{z}+1 \) types could be out of the market by the end of the period. To establish how many types could be out of the market by the end of period \( t \) agents ask themselves: “If I was a type-\( m \) agent, with \( m = t-\tau^* \), given the price history, how many values of \( t_0 \) would there be in my set \( \text{supp}(t_0 \mid I_{m,t}) \)?” The answer to this question is \( x_t \), a variable that plays a crucial role coordinating agents’ expectations, since given \( x_t \) and \( p' \), agents of all types know how to determine \( x_{t+1} \).

An example will be helpful to illustrate how \( x_t \) evolves as agents observe prices. Set \( \bar{z} = 2 \), (i.e. \( 2\bar{z} = 3/N \)), \( N = 10 \), \( t_0 = 20 \), and \( \tau^* = 10 \). Suppose that it is time \( t = 29 \), and that given an initial value \( x_1 \) and \( p^{28} = \{p_1,\ldots,p_{28}\} \), \( x_{29} = 3 \). (Later, I discuss how I choose a the initial value \( x_1 \), but for the purposes of this example, all that matters is that \( x_{29} = 3 \).)

---

28 For tractability reasons, I will restrict attention to trigger strategies.
According to (16), it is not yet time for type-20 agents to sell. However, type-20 agents are thinking about the possibility that type-19 agents may exist, and since $x_{29} = 3$, those hypothetical type-19 agents would have $\text{supp}(t_0 | I_{19,29}) = \{17,18,19\}$. This implies that, for type-20 agents, $\text{supp}(t_0 | I_{20,29}) = \{17,18,19,20\}$. Following (16), type-20 agents wait and see what $p_{29} / p_{28}$ reveals. Since nobody sells at $t = 29$, type-20 agents will find themselves in one of the following three scenarios, each of which will occur with probability 1/3:

1. If $c(p_{29} / p_{28}) = 0$, at time 30 everybody can see that all types are still in the market. For type-20 agents, this reveals that $t_0$ is 20, or, in other words, that $\text{supp}(t_0 | I_{20,30}) = \{20\}$. This also implies that $x_{30} = 1$. Type-20 agents do not sell at 30. Similarly, at $t = 31$, regardless of whether $c(p_{30} / p_{29})$ is 0, 1, or 2, neither type-20 nor type-21 agents sell, because type-21 agents will have $\text{supp}(t_0 | I_{21,31}) = \{20,21\}$. This implies that $x_{31} = 2$, and thus, that type-21 agents know that three types cannot possibly be out of the market by the end of time 31. At $t = 32$, regardless of whether $c(p_{31} / p_{30})$ is 0, 1, or 2, type-22 agents will have $\text{supp}(t_0 | I_{21,31}) = \{20,21,22\}$, implying $x_{32} = 3$, and making types 20, 21 and 22 sell.

2. If $c(p_{29} / p_{28}) = 1$, at $t = 30$, all agents understand that, at $t = 29$, either one type sold, or nobody sold. This implies that $\text{supp}(t_0 | I_{20,30}) = \{19,20\}$, and that $x_{30} = 2$. Since 3 types cannot possibly be out of the market by the end of period 30, type-20 agents do not sell at this time. Two possibilities may follow

   a. If $c(p_{30} / p_{29}) = 0$, at $t = 31$, all agents learn that nobody sold at time 29. Thus, type-20 agents learn they were first to observe the signal, and type-21 agents have $\text{supp}(t_0 | I_{21,31}) = \{20,21\}$. Since $x_{31} = 2$, this is the same as in scenario 1 at $t = 31$: Nobody sells at $t = 31$, and types 20, 21 and 22 sell at $t = 32$.

   b. If $c(p_{30} / p_{29}) = 1$ or $c(p_{30} / p_{29}) = 2$, at the beginning of period 31, type-20 agents are still not sure whether $t_0$ is 19 or 20, and type-31 agents have
supp(t₀ | I_{21,31})=\{19,20,21\}. Thus, x_{31} = 3, and types 20 and 21 sell at t = 31. With probability 2/3, this causes a crash at t = 32, and with probability 1/3, c(p_{31} / p_{30}) is 2, x_{32} = 3, and type-22 agents sell at t = 32, and the crash happens at t = 33.

3. If c(p_{29} / p_{28}) = 2, at time 30, it looks like 0, 1, or 2 types may have sold at time 29. For type-20 agents, this implies that supp(t₀ | I_{20,30})=\{18,19,20\}, and thus, x_{30} = 1. Type-20 agents sell at time 30, and from here, the possible continuations are:

a. If c(p_{30} / p_{29}) = 1, type-20 agents learn that t₀ = 20, and type-21 agents have supp(t₀ | I_{21,31})=\{20,21\}. Since this implies x_{31} = 2, type-21 agents will not sell at time 31. Thus, there will be a crash at time 32 with probability 1/3 (since type 20 sold at time 30 and never re-bought, c(p_{31} / p_{30}) can be 3), while with probability 2/3, c(p_{31} / p_{30}) ∈ \{1,2\} implying x_{32} = 3, triggering sales by types 21 and 22 at t = 32, and causing a crash at t = 33.

b. If c(p_{30} / p_{29}) = 2, x_{31} = 3, and type-21 agents sell at time 31. In turn, this will cause a crash at time 32 with probability 2/3, while with probability 1/3, c(p_{31} / p_{30}) will be 2, x_{32} will be 3, type-22 agents will sell at t = 32 and the bubble will burst at t = 33.

c. If c(p_{30} / p_{29}) = 3, the bubble bursts at time 31.

Having followed this example, we could re-write in a more precise way as follows:
Letting L(t) be the most recent time that x was z, and letting c(t) = \min_{t(t) ≤ s ≤ t} c(p_{s} / p_{s-1}) xₜ is simply given by

\[ xₜ = 1 + x_{t-1} - \max\{0, c(t) - c(p_{t-1} / p_{t})\} \].
gives an idea of how many people could have sold the last time. After that, \( x_{rel} = 1 + x_t \) if do not decrease, and otherwise. (Footnote: the more formal definition of equilibrium.)

Are agents willing to go along with this equilibrium? As before, it is necessary to verify that agents want to sell when (16) dictates that they should sell and that they want to wait when (16) says that they should wait.

[Describe algorithm.]

![Graph](image-url)

[To be completed]

6. Conclusion
I extend the model of bubbles by Abreu and Brunnermeier (2003) by allowing prices to reflect selling pressure monotonically and by introducing noise in order to keep prices from revealing all private information. Allowing for an amount of noise that is minimal, in the sense that it can conceal sales of one type but not more, I present a class of examples, where for appropriate parameter values, bubbles of arbitrary expected length arise in equilibrium. The assumption that there is a minimal amount of noise makes the model more tractable, but
is not particularly conducive to generating bubbles. As intuition suggests, adding more noise makes it easier to obtain bubble-equilibria, since examples suggest that the growth rate of bubble does not need to be as large as in the case with minimal noise. The bubbles presented in this paper are more robust than those in previous literature, since prices always depend on selling pressure, and the equilibria are robust to small changes in parameters.

From a normative standpoint, proponents of fundamentals-based models often argue that loosely formulated theories of bubbles are often invoked as justifications for various distortionary regulations. While this paper supports the plausibility of bubbles, it does not recommend regulating markets in order to preclude them. More research is needed to understand when and how bubbles are likely to arise and whether some form of regulation may improve efficiency in financial markets.

[To be completed]

6. References


Figure 1 --- Equilibrium with strategies given by (7)

Sales do not start until, at the earliest, time $t_0 + \tau^{**}$. At that time, type-$t_0$ agents sell if $p_{t_0 + \tau^{**} - 1} / p_{t_0 + \tau^{**} - 2}$ is medium and wait if it is high. They follow this sell-if-medium-wait-if-high rule for a total of $d$ periods, and if price ratios keep being high, in period $t_0 + \tau^*$, they sell even if $p_{t_0 + \tau^* - 1} / p_{t_0 + \tau^* - 2}$ is high. If $p_{t_0 + \tau^* - 1} / p_{t_0 + \tau^* - 2}$ is high, only type-$t_0$ agents sell at $t_0 + \tau^*$, whereas if $p_{t_0 + \tau^{**} - 1} / p_{t_0 + \tau^{**} - 2}$ is medium, all of the $d + 1$ types in the set \{ $t_0, t_0 + 1, \ldots, t_0 + d$ \} sell at that time.
APPENDIX A — Derivation of $\Gamma$

Start with (6) for $\tau^* = 1$, set $x = G / R$ and perform algebra steps as follows

\begin{align*}
2 > x^2 + x &\iff 2x^2 > 1 + x^3 \iff 0 > 1 + x^3 - 2x^2 \iff 0 > 1 + x(x^2 - 2x) \\
&\iff 0 > 1 + x(x^2 - 2x + 1 - 1) \iff 0 > 1 + x(x^2 - 2x + 1) - x \iff x < x(1 - x)(x - 1)
\end{align*}

Clearly, $x = 1$ is a root of the polynomial $1 + x^3 - 2x^2$ and for $x \neq 1$, we have

$$1 > x(x-1) \iff 0 > x^2 - x - 1 \iff \frac{1 - \sqrt{5}}{2} < x < \frac{1 + \sqrt{5}}{2}.$$ 

Thus, the other two roots are $(1 - \sqrt{5})/2$, and $\Gamma = (1 + \sqrt{5})/2$.

APPENDIX B — Proof of Lemmas 3 and 5

**Proof of Lemma 3**

To show that type-$n$ agents do not wish to sell at time $t = n + \tau^* - j$ (with $j \geq 1$) if $p_{t-1}/p_{t-2}$ is high, first note that the incentive for such preemptive sales comes from the possibility that $t_0$ could be $n - j$. If $t_0 = n - j$, type-$t_0$ agents would sell at time $t$ and, with probability $\pi$, these sales would precipitate a crash at $t + 1$. If $t_0 = n - j$ can be ruled out, type-$n$ agents know that nobody is selling at $t$, and have no incentive to sell because they know they can sell for a higher expected price at $t + 1$. Price histories for which $t_0 = n - j$ is either impossible or a zero-probability event can be divided into the following groups:

(I) If at least one of the price ratios $p_{t-2}/p_{t-3}, \ldots, p_{t-(d+1)}/p_{t-(d+2)}$ is medium, $t_0 = n - j$ is inconsistent with $p_{t-1}/p_{t-2}$ being high. To see this, observe that, if $p_{t-i}/p_{t-(i+1)}$ is medium for some $i \in [2, \ldots, d + 1]$ and $t_0$ is $n - j$, type-$t_0$ agents would have sold at $t - i + 1$, and $p_{t-1}/p_{t-2}$ could not possibly be high. In fact, only if $j = 1$ and $i = 2$, $p_{t-1}/p_{t-2}$ could be medium. Otherwise, the bubble would have burst before time $t$, or $p_{t-1}/p_{t-2}$ would be low, precipitating the crash at $t$.

(II) Even if all of the $d + 1$ ratios $p_{t-1}/p_{t-2}, \ldots, p_{t-(d+1)}/p_{t-(d+2)}$ are high, if $j > N - 1$, it is impossible that $t_0$ is $n - j$, since type-$n$ agents know that $t_0$ cannot be smaller than $n - (N - 1)$.

(III) If all of the $d + 1$ ratios $p_{t-1}/p_{t-2}, \ldots, p_{t-(d+1)}/p_{t-(d+2)}$ are high, and $j \leq N - 1$, $t_0$ could be $n - j$. But if $j > \tau^*$, type-$n$ agents have not observed the signal, and thus, since $\lambda \approx 0$, they assign zero probability to the event that $t_0 = n - j$.

Thus, a type-$n$ agent’s sell-or-wait tradeoff at time $t = n + \tau^* - j$ is nontrivial if all of the $d + 1$ price ratios $p_{t-1}/p_{t-2}, \ldots, p_{t-(d+1)}/p_{t-(d+2)}$ are high, $j \leq N - 1$ and $j \leq \tau^*$. To understand the problem faced by a type-$n$ agent in these cases, it is useful to consider the sell-or-wait trade-off for $\tau^* \geq 1$ and $j = 1$. Our type-$n$ agent knows that, with probability $\frac{1}{2}$, her type was second to observe the signal, i.e., $t_0 = n - 1$, and with probability $\frac{1}{2}$, her type was first, i.e., $t_0 = n$. If her type was second, the first type is selling at $t$. This will lead to a
low \( p_t / p_{t-1} \) and a crash at \( t+1 \) with probability \( \pi \), and with probability \( 1-\pi \) to a medium \( p_t / p_{t-1} \), in which case—given the assumption that \( d + 2 < \kappa N \)—she will be able to sell at \( t+1 \) having earned a high rate of return for one more period. If her type was first, nobody is selling at \( t \), and she will, for sure, sell at \( t+1 \) after one more period of price growth. Thus, for \( j = 1 \), waiting is optimal if inequality (9) holds.

Fortunately, even though (9) captures the simple case with \( j = 1 \), it is sufficient to rule out preemptive sales for arbitrary \( j \geq 1 \). To see why, consider a more general scenario with \( t = n + \tau^* - j \), \( d + 2 < \kappa N \), \( j \leq \min\{\tau^*, N-1\} \), and high ratios \( p_{t-i} / p_{t-(i+1)} \) for all \( i \in \{1, 2, \ldots, d+1\} \). Further, let \( \theta_j \) denote the expected discounted payoff that a type-\( n \) agent obtains if she follows (7) at all times (e.g., \( \theta_1 \) is the right hand-side of (9)). To show that \( \theta_1 > 1 \) implies \( \theta_j > 1 \) for all \( j > 1 \), I will first show that \( \theta_1 > 1 \) implies \( \theta_j > 1 \) and then generalize. If \( t = n + \tau^* - 2 \), there are three values of \( t_0 \) that type-\( n \) agents cannot rule out, \( n-2 \), \( n-1 \), and \( n \). If \( t_0 \) is \( n-2 \), type-\( t_0 \) agents sell at time \( t \), \( p_t / p_{t-1} \) will be low with probability \( \pi \) and medium with probability \( 1-\pi \). If \( t_0 \) is not \( n-2 \), \( p_t / p_{t-1} \) will be medium with probability \( 1-\pi \) and high with probability \( \pi \). In the former case, (7) dictates that type-\( n \) agents sell at \( t+1 \), having gained one more period of appreciation. In the latter case, type-\( n \) agents will have the option to sell and enjoy that period of appreciation, but they can also keep waiting, which will be optimal, because they will find themselves in a situation with \( t+1 = n + \tau^* - 1 \) and all of the last \( d+1 \) price ratios being high, and by assumption, \( \theta_1 > 1 \). Formally, we have

\[
\theta_2 = \frac{1}{3} \left[ \pi \left( \frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{2}{3} \left\{ (1-\pi) \frac{G}{R} + \pi \left[ \frac{1}{2} \pi \left( \frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \left( \frac{G}{R} \right)^2 \right] + \frac{1}{2} \left( \frac{G}{R} \right)^2 \right\}.
\]

Factoring \( G/R \) out of the curly bracket and since \( \theta_1 \) is the right-hand side of (9), we have

\[
\theta_2 = \frac{1}{3} \left[ \pi \left( \frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{2}{3} \left\{ (1-\pi) + \pi \theta_1 \right\} \geq \frac{1}{3} \left[ \pi \left( \frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{2}{3} \left( \frac{G}{R} \right)^2 = \frac{G}{R} \left( 1-\pi \right) + \frac{\pi}{3} \left( \frac{G}{R} \right)^{-(\tau^*+1)} \geq \theta_1 > 1.
\]

Generalizing to arbitrary \( j \), the support of \( t_0 \) contains the \( j+1 \) values \( \{n-j, \ldots, n\} \). If \( t_0 = n-j \), with probability \( \pi \) there will be a crash at \( t+1 \) and with probability \( 1-\pi \) the agent will gain one period of appreciation. If \( t_0 = n-k \), for \( k \in \{1, \ldots, j-1\} \), nobody sells at \( t \), type-\( n \) agents sell as soon as the price ratio becomes medium, and there is a chance that the bubble will burst before they sell, since if price ratios keep being high, at time

\[29\] Assuming \( d \geq 1 \). The possibility that \( j > d+1 \) is discussed later in the proof.
$n-k+\tau^*$, agents of type $t_0=n-k$ will sell, precipitating a crash in the next period with probability $\pi$. Finally, if $t_0=n$, nobody sells before type $n$, and type-$n$ agents will reap appreciation gains for up to $j$ periods. Thus, for general values of $j$, $\theta_j$ is given by

$$
\theta_j = \frac{1}{j+1} \left[ \frac{\pi}{G \backslash R} \frac{(G^j)^{-(\tau^*)}}{1} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \left[ \frac{G}{R} \frac{(\tau^*)}{1} + (1-\pi) \frac{G}{R} \right] + 
$$

$$
\quad + \cdots + \frac{1}{j+1} \left[ (1-\pi) \frac{G}{R} \frac{(\tau^*)}{1} + (1-\pi) \frac{G}{R} \right] + \pi \left[ (1-\pi) \frac{G}{R} \frac{(\tau^*)}{j-1} \right] + \pi \left[ (1-\pi) \frac{G}{R} \frac{(\tau^*)}{j} \right],
$$

which can be rewritten as

$$
\theta_j = \frac{1}{j+1} \left[ \frac{\pi}{G \backslash R} \frac{(G^j)^{-(\tau^*)}}{1} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \left[ \frac{G}{R} \frac{(\tau^*)}{1} + (1-\pi) \frac{G}{R} \right] + 
$$

$$
\quad + \frac{j}{j+1} \frac{\pi}{G \backslash R} \frac{(G^j)^{-(\tau^*)}}{1} + (1-\pi) \frac{G}{R} + \cdots + (1-\pi) \frac{G}{R} \frac{(\tau^*)}{j-1} + \frac{\pi}{G \backslash R} \frac{(\tau^*)}{j},
$$

Substituting $\theta_{j-1}$ for its value yields

$$
\theta_j = \frac{1}{j+1} \left[ \frac{\pi}{G \backslash R} \frac{(G^j)^{-(\tau^*)}}{1} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \frac{G}{R} \frac{(1-\pi)+\pi\theta_{j-1}}{R} \quad (17)
$$

And using the fact that $\theta_{j-1}>1$ we conclude that

$$
\theta_j > \frac{1}{j+1} \left[ \frac{\pi}{G \backslash R} \frac{(G^j)^{-(\tau^*)}}{1} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \frac{G}{R} \frac{(1-\pi)+\pi\theta_{j-1}}{R} > \theta_i > 1.
$$

I have implicitly assumed $j \leq d+1$. This is innocuous, since if $j > d+1$, type-$n$ agents will not sell at time $t+1$ even if $p_t / p_{t+1}$ is medium. However, since they have option to sell in that circumstance, their expected payoff cannot be smaller than if they sold then. Therefore, they have no incentive to sell preemptively at $t$.

Finally, note that the assumption that $d+2 < \kappa N$ is crucial, since (9) is correct only if, when $t_0=n-1$ and $p_t / p_{t+1}$ is medium, sales of type $t_0$ plus the $d+1$ types that sell at $t+1$ do not surpass $\kappa$. Otherwise, type-$n$ agents refuse to wait until $n+\tau^*$ to sell. If $d+1 \leq \kappa N < d+2$, medium prices burst the bubble if and only if $t_0=n-1$, and thus (9) would have to be replaced by (6), which fails for all $\tau^* \geq 1$. If $d+1 > \kappa N$ agents are even less willing to wait, since the probability of a crash increases further from $1/2$ to $1-\pi/2$.■

**Proof of Lemma 5**

First, I show that preemptive sales are most tempting when $t=n+\tau^{**}-1$, $\tau^{**} \geq 1$, $p_{t-1} / p_{t-2}$ is medium, and the $d+1$ (or more) price ratios immediately prior to $p_{t-1} / p_{t-2}$ are
all high. For these price histories, \( \text{supp}(t_0 \mid I_{n,t}) \) contains the \( d+3 \) elements \( \{n-(d+2), \ldots, n\} \). Three scenarios are possible:

I. **More than one type will be out of the market by the end of time \( t \), guaranteeing a crash at \( t+1 \).**
   a. If \( t_0 = n-(d+2) \), type-\( t_0 \) agents sold in period \( t-1 = t_0 + \tau^* \), \( \varepsilon_{r-1} \) was high enough to make \( p_{r-1} / p_{r-2} \) intermediate, and \( d+1 \) types \( n-(d+1), \ldots, n-1 \) will sell at \( t \).
   b. If \( t_0 = n-l \), for \( l \in \{2, \ldots, d+1\} \), nobody sold at \( t-1 \), but a low \( \varepsilon_{r-1} \) made \( p_{r-1} / p_{r-2} \) intermediate and will trigger sales by the \( l \) types \( n-l, \ldots, n-1 \).

II. **Only type \( n-1 \) will sell at \( t \), and the bubble will burst at time \( t+1 \) with probability \( \pi \).**

III. **Nobody will sell at time \( t \), and the bubble will continue to grow, at least for one more period and at most for \( d+2 \) more periods.** This happens if \( t_0 = n \).

In I and II, selling is preferable to waiting, and the opposite is true in case III. Next, I argue that, when \( \text{supp}(t_0 \mid I_{n,t}) = \{n-(d+2), \ldots, n\} \), the probability of case III is smallest. I do this by considering, in turn, what happens when \( \tau^* < 1 \), \( t = n + \tau^* - j \), for \( j \geq 2 \), or at least one of the \( d+1 \) price ratios immediately prior to \( p_{r-1} / p_{r-2} \) is intermediate.

a. \( \tau^* \geq 1 \), implies \( t \geq n \), i.e. that type-\( n \) agents have observed their signal at time \( t = n + \tau^* - 1 \). If \( t < n \), type-\( n \) agents have no significant incentive to sell preemptively, because \( \text{supp}(t_0 \mid I_{n,t}) = \{n-(d+2), n-(d+1), \ldots\} \), and since \( \lambda \approx 0 \), all values are roughly equiprobable. Thus, cases I and II have probability zero, and the bubble will continue to grow with probability one.

b. If \( t = n + \tau^* - j \), for \( j \geq 2 \), \( \text{supp}(t_0 \mid I_{n,t}) = \{n-(d+j+1), \ldots, n-1, n\} \). Cases I and II occur for the same number of values of \( t_0 \) as before, but the probability that nobody will sell at \( t \) grows, since case III applies to all \( t_0 \in \{n-(j-1), n\} \).

c. If the number of consecutive high ratios preceding \( p_{r-1} / p_{r-2} \) is \( k < d+1 \), \( \text{supp}(t_0 \mid I_{n,t}) = \{n-(k+1), \ldots, n\} \). Case I(a) is no longer possible, and if \( k < d \), the number of possible values of \( t_0 \) for which case I(b) arises is reduced. Thus, the probability of III increases.

Having argued that incentives to sell are strongest when \( \text{supp}(t_0 \mid I_{n,t}) = \{n-(d+2), \ldots, n\} \), it follows that, if preemptive sales are ruled out in that case, they are also ruled out in all other scenarios with \( t < n + \tau^* \) and intermediate \( p_{r-1} / p_{r-2} \). If \( \text{supp}(t_0 \mid I_{n,t}) = \{n-(d+2), \ldots, n\} \), as discussed in the text, waiting is preferable to selling if (11) holds. The expected gain \( W_d \) on the right-hand side of (11) is the average of all payoffs—weighted by their probability—that are possible if \( n = t_0 \) and all agents follow equilibrium strategies. If \( n = t_0 \), nobody sells at \( t \), \( p_t / p_{t-1} \) is medium with probability \( 1-\pi \) (triggering sales of type \( t_0 \) at \( t+1 \)) and high
with probability $\pi$, in which case type-$t_0$ agents wait until next period, and again, sell if the price is medium and wait if it is high. Continuing in this fashion, type-$n$ agents can benefit from up to $d+1$ periods of appreciation. Thus, $W_d$ is given by

$$W_d = (1-\pi) \frac{G}{R} + \pi \left( (1-\pi) \left( \frac{G}{R} \right)^2 + \pi \left( (1-\pi) \left( \frac{G}{R} \right)^{d+1} + \pi \left( \frac{G}{R} \right)^{d+1} \right) \right),$$

which, rearranging terms, can be written as (12). To see when (11) is satisfied, note that, holding $d$ constant, the right-hand-side is a decreasing function of $\tau^\star$, implying that, if (11) holds in the limit as $\tau^\star$ approaches infinity, it will also hold for lower values of $\tau^\star$. As $\tau^\star$ becomes arbitrarily large, (11) approximates

$$1 < \frac{(1-\pi) \left( \frac{G}{R} \right)^{d+1} + W_d}{d+3},$$

which, if $\pi G / R \neq 1$, is the same as

$$d + 3 < (1-\pi) \left( \frac{G}{R} \right)^{d+1} \left( 1 + \frac{\frac{\pi G}{R}}{\frac{\pi G}{R} - 1} - 1 \right) + \left( \frac{\pi G}{R} \right)^{d+1}.$$

If $\pi G / R > 1$, as $d$ increases, the left hand side increases linearly, while the right hand side grows exponentially. This implies existence of $\overline{d}$ such that whenever $d \geq \overline{d}$, (11) holds.

$\pi G / R > 1$ turns out to be not only sufficient, but also necessary for (11) to hold. If $\pi G / R \leq 1$, (11) fails for all $\tau^\star$ and all $d$. In that case, $\tau^\star$ can only be zero, meaning that $d = \tau^\star$ and thus that (9) applies only to $\tau^\star < \kappa N - 2$. Thus, if $\pi G / R \leq 1$, and agents play strategies given by (7), the overvaluation is corrected before (or at the latest at the same time as) a mass $\kappa$ of rational agents become aware of it, and thus, according to the definition by AB, bubbles do not arise. ■

**Appendix C – Details of Section 4.3**

In this appendix I show that, even after reentry is allowed, (9) is still sufficient to rule out sales by type-$n$ agents at time $t = n + \tau^\star - j$ for $j \geq 1$, if $p_{t-1} / p_{t-2}$ is high. It is only necessary to consider cases where all of the $d+1$ price ratios $p_{t-1} / p_{t-2}, \ldots, p_{t-(d+1)} / p_{t-(d+2)}$ are high, since, as discussed in the proof of lemma 3, if this does not hold, there is no incentive to sell preemptively.

Let us thus examine the decision of a type-$n$ agent $t = n + \tau^\star - j$ with $j \geq 1$, and all of $d+1$ price ratios before $t$ being high. If $j = 1$, and the type-$n$ agent sold preemptively at $t$, she would not reenter at $t+1$ even if $p_t / p_{t-1}$ was high, since other type-$n$ agents would be selling and by (8), it would be better to stay out of the market. If $j > 1$, a type-$n$ agent who had sold preemptively at $t$ would reenter at $t+1$ if $p_t / p_{t-1}$ was high, since reentering would

---

30 Proof of this claim is available from the author upon request.
yield an expected payoff \( \theta_{j-1} > 1 \), where, as in the proof of lemma 3, \( \theta_j \) denotes the expected discounted payoff that a type-\( n \) agent obtains if she follows (7) at all times. However, even though selling preemptively and reentering is preferable to selling preemptively without the option to reenter, it is still better to not sell preemptively in the first place. To see this, note that \( \theta_j \) exceeds the payoff from exiting and reentering if

\[
\theta_j > \frac{1}{j+1} + \frac{j}{j+1}[\frac{1}{1-\pi} + \pi \theta_{j-1}].
\]

Recalling (9), we have

\[
\frac{1}{j+1} \left[ \pi \left( \frac{G}{R} \right)^{(r^{*+1})} + \left(1-\pi\right) \frac{G}{R} \right] + \frac{j}{j+1} \left(1-\pi\right) + \pi \theta_{j-1} > \frac{1}{j+1} + \frac{j}{j+1}[\frac{1}{1-\pi} + \pi \theta_{j-1}].
\]

Simplifying, and rearranging terms, we get

\[
\left[ \pi \left( \frac{G}{R} \right)^{(r^{*+1})} + \left(1-\pi\right) \frac{G}{R} \right] + \frac{j}{j+1} \left(1-\pi\right) + \pi \theta_{j-1} > 1
\]

Using the fact that \( \theta_j > 1 \) for all \( j \geq 1 \), we have

\[
\left[ \pi \left( \frac{G}{R} \right)^{(r^{*+1})} + \left(1-\pi\right) \frac{G}{R} \right] + \frac{j}{j+1} \left(1-\pi\right) + \pi \theta_{j-1} > \left[ \pi \left( \frac{G}{R} \right)^{(r^{*+1})} + \left(1-\pi\right) \frac{G}{R} \right] + \frac{j}{j+1} \left(1-\pi\right)
\]

and the right-hand side is greater than one, since

\[
\left[ \pi \left( \frac{G}{R} \right)^{(r^{*+1})} + \left(1-\pi\right) \frac{G}{R} \right] + j > 1 + j
\]

\[
\Leftrightarrow \left(1 - \frac{\pi}{j+1}\right) \frac{G}{R} + \frac{\pi}{j+1} \left( \frac{G}{R} \right)^{(r^{*+1})} > 0.1.
\]

In words, selling preemptively is preferable to waiting only if \( t_0 = n - j \). For all other values of \( t_0 \) selling preemptively (and reentering at \( t+1 \) if \( p_t / p_{t+1} \) is high) implies missing out on a fraction of the expected appreciation gains. If (9) holds, this opportunity cost is always large enough to deter preemptive sales. **Q.E.D.**