Optimal Nonlinear Income Taxation with Learning-by-Doing

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Abstract

This paper examines a two-period model of optimal nonlinear income taxation with learning-by-doing, in which second-period wages are an increasing function of first-period labour supply. We consider the cases when the government can and cannot commit to its second-period tax policy. In both cases, the canonical Mirrlees/Stiglitz results regarding optimal marginal tax rates no longer apply. In particular, if the government cannot commit and each consumer’s skill-type is revealed, it is optimal to distort the high-skill type’s labour supply downwards through a positive marginal tax rate to relax an incentive-compatibility constraint. Our analysis therefore identifies a setting in which a positive marginal tax rate on the highest-skill individual can be justified, despite its depressing effect on labour supply and wages.

Keywords: Income taxation, learning-by-doing, commitment.

JEL classifications: H21, H24.

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1 Introduction

In recent years, a literature known as the ‘new dynamic public finance’ has emerged that extends the static Mirrlees [1971] model of optimal nonlinear income taxation to a dynamic setting. For the most part, this literature assumes that random productivity shocks determine future wages, and that the government can commit to its future tax policy.\(^1\) In this paper, we assume that wages are determined by ‘learning-by-doing’, i.e., an individual who works longer in the present becomes more productive through work experience, and therefore enjoys higher wages in the future. Our interest in learning-by-doing stems from the observation that, while the role of education in raising wages has received a great deal of attention in terms of its implications for redistributive taxation, as far as we know the similar role of learning-by-doing has received no attention.\(^2\) Given that work experience is arguably at least as important as formal education in raising productivity in many occupations, the implications of learning-by-doing for redistributive taxation are potentially important. We consider the case when the government can commit to its future tax policy, but we also consider the case when the government cannot commit. While both cases are of interest, we think the no-commitment case is particularly relevant, since the second-best nature of the Mirrlees framework stems from the assumption that an individual’s skill type is private information. But taxation in earlier periods may result in this information being revealed to the government, which would enable the government to implement first-best (lump-sum) taxation in latter periods.\(^3\) As a result, some individuals may be reluctant to reveal their skill type in earlier periods, in order to avoid being subjected to first-best taxation in latter periods.

We work with the two-type version of the Mirrlees model with a continuum of consumers of each type, but extend it to a two-period setting. It is well known that in

\(^1\)See Golosov, et al. [2006] for a review of the ‘new dynamic public finance’ literature. This literature has been developed by macroeconomists who recognise that the representative-agent (Ramsey) approach to optimal taxation omits some important features that are relevant for determining optimal taxes.

\(^2\)Learning-by-doing has featured in growth models with taxation, but the focus is on how the government can set taxes to smooth the business cycle. For example, see Martin and Rogers [2000].

\(^3\)Indeed, one of the arguments made by the ‘new dynamic public finance’ literature against the representative-agent approach to optimal taxation is that it rules out via an ad hoc assumption the use of lump-sum taxes. Likewise, ruling out lump-sum taxes in a dynamic Mirrlees setting via a commitment assumption might also be considered inappropriate.
the static two-type model, a government with redistributive goals will impose a positive marginal tax rate on the low-skill type and a zero marginal tax rate on the high-skill type. The rationale for the positive marginal tax rate on the low-skill type is to distort her labour supply downwards to relax the high-skill type’s incentive-compatibility constraint. One might expect that learning-by-doing simply gives the government an additional motive for marginal distortions, e.g., distorting the low-skill type’s first-period labour supply upwards may facilitate redistribution by increasing her second-period wage. Or distorting both types first-period labour supply upwards may increase social welfare via higher second-period wages. However, the only motive the government has to implement marginal tax rate distortions remains that to relax the high-skill type’s incentive-compatibility constraint. This is because the consumers rationally consider the effect on their second-period wage when deciding their first-period labour supply. Thus the government has no reason to distort individual behaviour to correct any sort of dynamic inconsistency. Nevertheless, we show that the static optimal marginal tax rate results no longer apply, even when the government can commit to its second-period tax policy.

When the government can commit, it may be optimal for the low-skill type to face a negative marginal tax rate in the first period in order to relax the high-skill type’s incentive-compatibility constraint. This result also applies when the government cannot commit. Moreover, when the government cannot commit, the standard ‘no-distortion-at-the-top’ result no longer holds. If the consumers are completely separated in period 1, thus giving the government enough information to implement first-best taxation in period 2, it is optimal for the high-skill type to face a positive marginal tax rate in period 1. This is because the government wants to distort the high-skill type’s first-period labour supply downwards to relax an incentive-compatibility constraint. If the

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4 There is a large literature that works with the two-type version of the Mirrlees [1971] model introduced by Stiglitz [1982]. This is due to its simplicity, but also because theory alone sheds little light on the pattern of optimal marginal tax rates over the intermediate skills range. An exception is Boadway and Jacquet [2006], who show that some features of the entire optimal income tax schedule can be characterised theoretically if the government’s objective is a maxi-min social welfare function.

5 In general, if consumers exhibit dynamically inconsistent preferences, then a clear-cut case can be made for corrective taxation. For example, see O’Donoghue and Rabin [2006] for a time-inconsistency argument in favour of taxes on unhealthy foods.
consumers are completely pooled in period 1, thus constraining the government to use second-best taxation in period 2, the government wants to distort the high-skill type’s first-period labour supply upwards to relax an incentive-compatibility constraint. This suggests that the high-skill type should face a negative marginal tax rate in period 1, but due to some other factors at work the high-skill type’s first-period optimal marginal tax rate cannot be signed. We also consider the case when some, but not all, of the high-skill consumers are pooled with the low-skill consumers in period 1. In this case, the high-skill consumers who are separated face a positive marginal tax rate in period 1 (as in the complete separation case), while the marginal tax rate faced by those high-skill consumers who are pooled cannot be signed (as in the complete pooling case).

Since the second period of our model is the last period, the optimal tax problem in the second period is similar to that in a static model, and therefore the static results apply. The only exception is when first-best taxation is possible in the second period, in which case both types naturally face zero marginal tax rates. Our focus therefore is on optimal taxation in the first period, since it is the first period that captures the essential challenge of dynamic taxation. That is, when choosing its present tax policy, the government must also consider how its choice will affect its taxation possibilities in the future.

The present paper is related to recent work by Berliant and Ledyard [2005], Apps and Rees [2006], and Brett and Weymark [2008c]. These papers also employ two-period nonlinear income tax models in which the government cannot commit, although learning-by-doing does not feature in their models. Instead, they assume that wages are fixed and constant through time. In Berliant and Ledyard [2005] there is a continuum of types, and their focus is on deriving conditions under which the consumers are separated in the first period. They contrast this possibility with the infinite-horizon model of Roberts [1984], in which the consumers are never separated. In Apps and Rees [2006] and Brett

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6. Gaube [2007] also considers a two-period model of nonlinear income taxation without learning-by-doing, but assumes that the government can commit. His focus is on showing that, if the government cannot control consumption in each period due to ‘hidden’ savings, then the government cannot implement the optimal long-term tax contract with a pair of short-term tax contracts. This problem does not arise in our model, however, since we assume there are no savings.
and Weymark [2008c] there are only two types, but in Apps and Rees [2006] there is a continuum of consumers of each type while in Brett and Weymark [2008c] there is a single consumer of each type. This makes partial pooling possible in Apps and Rees [2006], while in Brett and Weymark [2008c] the consumers are either separated or pooled.\footnote{7} Our model is therefore most closely related to that of Apps and Rees [2006], although methodologically our analysis is closer to Brett and Weymark [2008c]. They show that the static optimal marginal tax rate results remain intact if the government can commit. The static results also remain intact in the first period if the government cannot commit and there is separation. However, if there is pooling, the high-skill type faces a positive marginal tax rate and the low-skill type faces a negative marginal tax rate in the first period.\footnote{8} Our analysis shows that these results no longer hold—and are often reversed—when wages are determined by learning-by-doing.

The remainder of the paper is organised as follows. Section 2 presents the key features of the model that we consider. Section 3 examines optimal income taxation with commitment, while Section 4 examines optimal income taxation without commitment. Section 5 contains some closing remarks, while proofs are relegated to an appendix.

## 2 The Economy

There is a continuum of consumers with unit measure, with a proportion $\phi \in (0, 1)$ being high-skill workers and the remaining $1 - \phi$ being low-skill workers. Both types of consumer live for two periods. Consumption by type $i$ ($i = 1, 2$) in period $t$ ($t = 1, 2$) is denoted by $c_t^i$, and labour supply by type $i$ in period $t$ is denoted by $l_t^i$. Type $i$'s wage is equal to $w^1_i$ in period 1 and $w^2_i = w^2_i(l^1_i)$ in period 2, where it is assumed that $\partial w^2_i(l^1_i)/\partial l^1_i > 0$ which captures the notion of learning-by-doing. That is, an increase in

\footnote{7}{The focus of Brett and Weymark [2008c] is on the desirability of nonlinear savings taxation; there are no savings in the Apps and Rees [2006] model.}

\footnote{8}{It should be noted that these distortions are not implemented to relax an incentive-compatibility constraint. When there is pooling, the government offers a single tax treatment which in effect is chosen based on an average of the high-skill and low-skill wage rates. This results in the high-skill type’s labour supply being distorted downwards, and the low-skill type’s labour supply being distorted upwards, to earn the same level of pre-tax income.}
type \(i\)'s labour supply in period 1 raises her productivity and hence wage rate in period 2. We assume throughout that \(w^1_2 > w^1_1\) and \(w^2_2(l^1_2) > w^2_1(l^1_1)\) for all relevant levels of \(l^1_2\) and \(l^1_1\), so type 1 consumers are low-skill workers and type 2 consumers are high-skill workers. Type \(i\)'s pre-tax income in period \(t\) is denoted by \(y^t_i = w^t_i l^t_i\).

The consumers have identical preferences over consumption and labour in each period, which are represented by the additively separable utility function \(u(c^t_i) - v(l^t_i)\), where \(u(\cdot)\) is increasing and strictly concave and \(v(\cdot)\) is increasing and strictly convex. To obtain an expression for the marginal tax rate faced by type \(i\) in period \(t\), suppose the consumers faced smooth nonlinear income tax functions \(T^1(y^1_i)\) and \(T^2(y^2_i)\) in periods 1 and 2, respectively. Then type \(i\)'s behaviour can be described by the following programme:

\[
\text{max} \quad c^1_i, l^1_i, c^2_i, l^2_i \quad \{u(c^1_i) - v(l^1_i) + u(c^2_i) - v(l^2_i) \mid c^1_i \leq y^1_i - T^1(y^1_i) \land c^2_i \leq y^2_i - T^2(y^2_i)\} \quad (2.1)
\]

where for simplicity it is assumed that there are no savings and future utility is not discounted. Thus each consumer’s lifetime utility is the simple sum of her utility in each period, and consumption in each period cannot exceed post-tax income in that period. It is shown in the appendix that the solution to programme (2.1) yields the following expressions for the marginal tax rates:

\[
MTR^1_i := \frac{\partial T^1(\cdot)}{\partial y^1_i} = 1 - \frac{v'(l^1_i)}{u'(c^1_i) w^1_i} + \frac{v'(l^2_i) l^2_i}{u'(c^1_i) w^1_i w^2_i} \frac{\partial w^2_i(\cdot)}{\partial l^1_i} \quad (2.2)
\]

\[
MTR^2_i := \frac{\partial T^2(\cdot)}{\partial y^2_i} = 1 - \frac{v'(l^2_i)}{u'(c^2_i) w^2_i} \quad (2.3)
\]

where \(MTR^t_i\) denotes the marginal tax rate faced by type \(i\) in period \(t\). Equation (2.3) shows that the marginal tax rate in period 2 is equal to one minus the marginal rate of substitution of pre-tax income for consumption in period 2, which is the same result as obtained in static models.\(^{10}\) This follows simply from the fact that period 2 is the

\(^9\)It is well known that in models with a finite number of types, the optimal income tax schedule may not be differentiable. Thus we follow the standard practice of deriving expressions for ‘implicit’ marginal tax rates in terms of derivatives of the utility function.

\(^{10}\)See, e.g., Stiglitz [1982], Weymark [1987], and Brett and Weymark [2008a, 2008b].
last period of our model. Equation (2.2), however, shows that the expression for the marginal tax rate in period 1 is complicated by the fact that a marginal increase in $y^1_i$, which necessitates a marginal increase in $l^1_i$, results in a *ceteris paribus* increase in utility in period 2 via higher wages. It is the last term in (2.2) that captures this effect. Thus in period 1 the slope of each type’s indifference curve is not tangent to the budget set at an optimum,\textsuperscript{11} meaning the first-period marginal tax rate is not simply equal to one minus the marginal rate of substitution as in static models.

### 3 Optimal Income Taxation with Commitment

If in period 1 the government can commit to its tax policy in period 2, the government cannot exploit any information that may be revealed in period 1 to redesign its second-period tax system. In this case, the government can be described as choosing a ‘tax contract’ $\langle c^1_1, y^1_1, c^2_1, y^2_1 \rangle$ for the low-skill type and $\langle c^1_2, y^1_2, c^2_2, y^2_2 \rangle$ for the high-skill type to maximise:\textsuperscript{12}

$$(1 - \phi) \left[ u(c^1_1) - v \left( \frac{y^1_1}{w^1_1} \right) + u(c^2_1) - v \left( \frac{y^2_1}{w^1_1} \right) \right] + \phi \left[ u(c^1_2) - v \left( \frac{y^1_2}{w^2_1} \right) + u(c^2_2) - v \left( \frac{y^2_2}{w^2_2} \right) \right]$$

subject to:

$$11\text{ We thank a referee for pointing this out.}$$

$$12\text{ A tax contract consists of pre-tax income and post-tax income (which is equal to consumption) in each period. The difference between pre-tax income and consumption is total taxes paid (or transfers received). While we do not observe such a tax system in practice, the ‘Revelation Principle’ implies that any tax system (or any mechanism) can be replicated by an incentive-compatible direct mechanism.}$$
where \( w^2_1 = w^2_1 \left( \frac{y^1_1}{w^1_1} \right) \), \( w^2_2 = w^2_2 \left( \frac{y^1_2}{w^1_2} \right) \), \( \tilde{w}^1_1 = w^1_1 \left( \frac{y^2_1}{w^2_1} \right) \), and \( \tilde{w}^2_2 = w^2_2 \left( \frac{y^1_1}{w^2_2} \right) \). The objective function (3.1) is a utilitarian social welfare function, where the utility functions have been written in terms of the government’s choice variables \( c^i_t \) and \( y^i_t \). Equations (3.2) and (3.3) are budget constraints requiring that total tax revenues be non-negative in each period.\(^{13}\) Equations (3.4) and (3.5) are incentive-compatibility constraints for type 1 and type 2 consumers, respectively. Following the standard practice, we assume that the government knows the distribution of types, but each individual consumer’s skill type is private information. The government must therefore satisfy the incentive-compatibility constraints to induce each type to choose their intended tax contract, rather than ‘mimicking’ the other type by choosing the other type’s tax contract. We focus on what Stiglitz [1982] calls the ‘normal’ case and what Guesnerie [1995] calls ‘redistributive equilibria’, in the sense that we assume the high-skill type’s incentive-compatibility constraint (3.5) binds at an optimum while the low-skill type’s incentive-compatibility constraint (3.4) is slack.\(^{14}\) Most of the literature has focused on this case, the rationale being that the government uses its taxation powers to redistribute from high-skill to low-skill consumers, which creates an incentive for high-skill consumers to mimic low-skill consumers, but not vice versa.

It is shown in the appendix that the solution to programme (3.1) – (3.5) yields:

**Proposition 1** Optimal income taxation with learning-by-doing and when the government can commit to its second-period tax policy is characterised by: \( MTR^1_1 \geq 0 \), \( MTR^1_1 = 0 \), \( MTR^2_1 > 0 \), and \( MTR^2_2 = 0 \).

The pattern of marginal tax rate distortions in the second period—namely, that type 1 consumers face a positive marginal tax rate and type 2 consumers faces a zero marginal tax rate—is the same as that in a static model. This is simply because period 2 is the last period of our model. Type 2 consumers also face a zero marginal tax rate in period 1, but type 1’s marginal tax rate cannot be signed. In particular, it is now possible that the government will want to distort type 1’s first-period labour

\(^{13}\)As with the consumers, for simplicity we do not permit the government to save.

\(^{14}\)We will continue to assume that the low-skill type’s incentive-compatibility constraint never binds; it is therefore omitted throughout the remainder of the paper.
supply upwards via a negative marginal tax rate to relax type 2’s incentive-compatibility constraint. There are two forces at work here. On the one hand, type 1 consumers work longer than type 2 consumers when both types choose to earn $y_1$ (since $w_1^1 < w_1^2$). Therefore, type 1 suffers a greater disutility from labour supply in period 1 than does a mimicking type 2. This gives the government the usual motive to distort type 1’s labour supply downwards via a positive marginal tax rate to deter mimicking. But on the other hand, learning-by-doing implies that second-period wages are increasing in first-period labour supply. Type 1 consumers may therefore obtain a greater increase in their second-period utility from their first-period labour supply than does a mimicking type 2. Accordingly, it is possible that the lifetime marginal disutility that type 1 consumers incur from additional first-period labour is less than that incurred by a mimicking type 2, even though a mimicking type 2 works less than type 1. If this is the case, the government can relax the incentive-compatibility constraint by distorting type 1’s first-period labour supply upwards through a negative marginal tax rate. It is also possible that the lifetime marginal disutilities that type 1 consumers and a mimicking type 2 incur from additional first-period labour are the same. In this case, type 1 consumers will face a zero marginal tax rate because distortions to their first-period labour supply will not relax the incentive-compatibility constraint.

4 Optimal Income Taxation without Commitment

If in the first period the government cannot commit to its second-period tax policy, each consumer knows that if she reveals her type in period 1 she will be subjected to personalised lump-sum taxation in period 2. This implies that a high-skill consumer must be offered a relatively attractive tax contract in period 1 to reveal her type, in order to compensate for the unfavourable tax treatment she will receive in period 2. From a social welfare point of view, the lack of redistribution in period 1 required to obtain type information may be too costly. Instead, the government may be better off pooling some or all of the high-skill consumers with the low-skill consumers so that type information is not revealed, even though it is then constrained to use second-best taxation in period
2. As Brett and Weymark [2008c] note, deciding whether the government is better off with a tax system that separates or pools in the first period requires a comparison of the maximised values of the social welfare function in each case. In general, such comparisons depend upon the exact form of the utility function and the distribution of wages. We therefore examine in turn the complete separation, complete pooling, and partial pooling cases.

4.1 Complete Separation in Period 1

If the tax system separates all the high-skill consumers from the low-skill consumers in the first period, the government’s behaviour in the second period can be described as follows. Choose \( h_1^c, y_1^c \) and \( h_2^c, y_2^c \) to maximise:

\[
(1 - \phi) \left[ u(c_1^2) - v \left( \frac{y_1^2}{w_1^2} \right) \right] + \phi \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) \right] \quad (4.1)
\]

subject to:

\[
(1 - \phi) [y_1^2 - c_1^2] + \phi [y_2^2 - c_2^2] \geq 0 \quad (4.2)
\]

where (4.1) is social welfare in period 2, and (4.2) is the second-period budget constraint.

Since the government can identify the consumers, type 1 consumers must accept \( h_1^c, y_1^c \) and type 2 consumers must accept \( h_2^c, y_2^c \). That is, the government is not constrained by incentive-compatibility constraints.

Since \( w_1^2 = w_1^2 \left( \frac{y_1}{w_1^1} \right) \) and \( w_2^2 = w_2^2 \left( \frac{y_2}{w_2^1} \right) \), the solution to the above programme yields the functions \( c_1^2(\phi, y_1^1, w_1^1, y_1^2, w_1^2), c_2^2(\phi, y_1^1, w_1^1, y_1^2, w_1^2) \) and \( y_2^2(\phi, y_1^1, w_1^1, y_1^2, w_1^2) \).

Substituting these into (4.1) yields the value function \( W^2(\phi, y_1^1, w_1^1, y_1^2, w_1^2) \).

The consumers and the government know that, if there is complete separation in period 1, the government will solve programme (4.1) – (4.2) in period 2. Therefore, the government in period 1 can be described as choosing \( c_1^1, y_1^1 \) and \( c_2^1, y_2^1 \) to maximise:

\[
(1 - \phi) \left[ u(c_1^1) - v \left( \frac{y_1^1}{w_1^1} \right) \right] + \phi \left[ u(c_2^1) - v \left( \frac{y_2^1}{w_2^1} \right) \right] + W^2(\cdot) \quad (4.3)
\]

subject to:

\[
(1 - \phi) [y_1^1 - c_1^1] + \phi [y_2^1 - c_2^1] \geq 0 \quad (4.4)
\]
\begin{equation}
u(c_2^1) - v \left( \frac{y_2^1}{w_2^1} \right) - u(c_2^2(\cdot)) - v \left( \frac{y_2^2(\cdot)}{w_2^2} \right) \geq u(c_1^1) - v \left( \frac{y_1^1}{w_1^1} \right) + u(c_1^2(\cdot)) - v \left( \frac{y_1^2(\cdot)}{w_2^2} \right) \end{equation}

The government chooses \(\langle c_1^1, y_1^1 \rangle\) and \(\langle c_2^1, y_2^1 \rangle\) while taking into account how its choice will affect social welfare in period 2. Its first-period objective function (4.3) therefore includes the second-period value function \(W^2(\cdot)\). Equation (4.4) is the first-period budget constraint, while (4.5) is type 2’s incentive-compatibility constraint. In order for a type 2 consumer to be willing to reveal her type in period 1, the utility she obtains from \(\langle c_2^1, y_2^1 \rangle\) in period 1 plus the utility she obtains from the first-best tax contract \(\langle c_2^2(\cdot), y_2^2(\cdot) \rangle\) that she must accept in period 2 has to be greater than or equal to the utility she could obtain from \(\langle c_1^1, y_1^1 \rangle\) in period 1 plus the utility from the low-skill type’s first-best tax contract \(\langle c_1^2(\cdot), y_1^2(\cdot) \rangle\) that she would receive in period 2. That is, if a type 2 consumer chooses \(\langle c_1^1, y_1^1 \rangle\) in period 1, she is announcing to the government that she is low-skill and will therefore be treated as such in the second period.\(^{15}\)

It is shown in the appendix that the solutions to programmes (4.1) – (4.2) and (4.3) – (4.5) together imply:

**Proposition 2** Optimal income taxation with learning-by-doing, when the government cannot commit to its second-period tax policy, and when the consumers are completely separated in the first period is characterised by: \(\text{MTR}_1^1 \geq 0\), \(\text{MTR}_2^1 > 0\), \(\text{MTR}_1^2 = 0\), and \(\text{MTR}_2^2 = 0\).

The zero marginal tax rate faced by both types in period 2 follows simply from the first-best nature of taxation in that period. What is more interesting is the pattern of marginal tax rate distortions in period 1. In particular, the high-skill type necessarily faces a positive marginal tax rate. The reason is as follows. The first-best allocation in period 2 involves both types receiving the same level of consumption, but type 2 consumers work longer than type 1 consumers.\(^{16}\) Therefore, type 2 consumers obtain a lower level of utility than type 1 consumers in the second period. Indeed, it can be shown that for the many-type case, first-best taxation has utility decreasing in wages, since all

\(^{15}\)Our assumption that there is a continuum of consumers plays a role here, since it implies that each consumer has a zero mass. Therefore, if a high-skill consumer pretends to be low-skill, the distribution of types is not affected.

\(^{16}\)Using equations (A.24) and (A.26) in the appendix we obtain \(u'(c_1^1) = u'(c_2^2)\) which implies that \(c_1^2 = c_2^2\). Using (A.25) and (A.27) we obtain \(v'(l_1^2)/v'(l_2^2) = w_1^2/w_2^2\) which implies that \(l_1^2 < l_2^2\).
types receive the same level of consumption but labour supply is increasing in skill type. By distorting type 2’s labour supply downwards in the first period, the government is decreasing their second-period wage, but actually increasing their second-period utility. Type 2’s consumption in the second period falls, but their labour supply falls by more, resulting in a net increase in utility. Moreover, type 1’s consumption falls and their labour supply increases, resulting in lower utility for type 1 consumers which makes mimicking less attractive. Thus distorting type 2’s labour supply downwards in the first period makes them more willing to reveal their type, i.e., the incentive-compatibility constraint is relaxed.

The sign of the marginal tax rate faced by type 1 consumers in period 1 is ambiguous. This is for the same reasons as to why it is ambiguous when the government can commit (see Section 3), but now there are some additional complications. On the one hand, distorting type 1’s first-period labour supply upwards raises their second-period wage. This reduces the extent of redistribution undertaken using first-best taxation in period 2, which makes type 2 consumers better off and thereby relaxes the incentive-compatibility constraint. On the other hand, distorting type 1’s first-period labour supply upwards raises their own second-period consumption and labour supply. While it can be shown that this makes type 1 consumers worse off in period 2 as their consumption rises by less than their labour supply, this is not necessarily the case for a mimicking type 2 consumer since they have a higher wage. Therefore, a mimicking type 2 consumer’s consumption in period 2 may rise by more or less than their required labour supply, rendering the net effect on their second-period utility ambiguous. Thus these additional factors further serve to make type 1’s first-period marginal tax rate ambiguous.

4.2 Complete Pooling in Period 1

If the consumers are completely pooled in the first period, the government’s behaviour in the second period can be described as follows. Choose \((c_1^2, y_1^2)\) and \((c_2^2, y_2^2)\) to maximise:

\[
(1 - \phi) \left[ u(c_1^2) - v \left( \frac{y_1^2}{w_1^2} \right) \right] + \phi \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) \right]
\]

\[(4.6)\]
subject to:

\begin{equation}
(1 - \phi) \left[ y_1^2 - c_1^2 \right] + \phi \left[ y_2^2 - c_2^2 \right] \geq 0 \tag{4.7}
\end{equation}

\begin{equation}
u(c_2) - v \left( \frac{y_2^2}{w_2^2} \right) \geq u(c_1^2) - v \left( \frac{y_1^2}{w_1^2} \right) \tag{4.8}
\end{equation}

where (4.6) is social welfare in period 2, (4.7) is the second-period budget constraint, and (4.8) is type 2’s incentive-compatibility constraint. Since \( w_1^2 = w_1 \left( \frac{y_1}{w_1^2} \right) \) and \( w_2^2 = w_2 \left( \frac{y_1}{w_2^2} \right) \), the solution to the above programme yields the functions \( c_1^2(\phi, y_1, w_1^1, w_1^2), y_1^4(\phi, y_1, w_1^1, w_1^2), c_2^2(\phi, y_1, w_1^1, w_1^2) \) and \( y_2^2(\phi, y_1, w_1^1, w_1^2) \), where \( y_1 \) denotes the pre-tax income earned by both types in the first period. Substituting these functions into (4.6) yields the value function \( W^2(\phi, y_1, w_1^1, w_1^2) \).

The consumers and the government know that, if there is complete pooling in period 1, the government will solve programme (4.6) – (4.8) in period 2. Therefore, the government in period 1 can be described as choosing \( h_{c_1, y_1} \) to maximise:

\begin{equation}
(1 - \phi) \left[ u(c_1) - v \left( \frac{y_1^1}{w_1^1} \right) \right] + \phi \left[ u(c_1) - v \left( \frac{y_1^1}{w_1^2} \right) \right] + W^2(\cdot) \tag{4.9}
\end{equation}

subject to:

\begin{equation}
y_1^1 - c_1^1 \geq 0 \tag{4.10}
\end{equation}

where \( \langle c_1^1, y_1^1 \rangle \) is the tax contract offered to both types in period 1. When choosing \( \langle c_1^1, y_1^1 \rangle \), the government considers how its choice will affect social welfare in period 2. Its first-period objective function (4.9) therefore includes the second-period value function \( W^2(\cdot) \). Equation (4.10) is the first-period budget constraint. As all consumers are offered the single choice of \( \langle c_1^1, y_1^1 \rangle \), the government does not face an incentive-compatibility constraint in the first period.\[^1\]

It is shown in the appendix that the solutions to programmes (4.6) – (4.8) and (4.9) – (4.10) together imply:

**Proposition 3** Optimal income taxation with learning-by-doing, when the government cannot commit to its second-period tax policy, and when the consumers are completely

\[^1\] Although the government does face the second-period incentive-compatibility constraint (4.8) indirectly through the value function \( W^2(\cdot) \).
pooled in the first period is characterised by: \( MTR_1^1 \geq 0, MTR_2^1 \geq 0, MTR_1^2 > 0, \) and \( MTR_2^2 = 0 \). Moreover, \( (1 - \phi)MTR_1^1 + \phi MTR_1^2 < 0 \) which implies that \( MTR_1^1 < 0 \) and/or \( MTR_2^1 < 0 \).

When there is complete pooling in the first period, the second-period optimal tax problem is identical to that in a static model. Hence the usual pattern of marginal tax rate distortions is obtained in period 2. To understand why the marginal tax rates faced by both types in period 1 are ambiguous, suppose learning-by-doing was absent from the model. Then without taxation, type 2 consumers would choose to earn a higher income than type 1 consumers (as both types have the same preferences, but \( w_2^1 > w_1^1 \)). When both types are subjected to the same tax treatment in period 1, the government in effect chooses \( y_1 \) based on an average of \( w_1^1 \) and \( w_2^1 \). This results in type 1’s labour supply being distorted upwards to earn \( y_1 \) and type 2’s labour supply being distorted downwards to earn \( y_1 \). Therefore, without learning-by-doing, type 1 consumers would face a negative marginal tax rate and type 2 consumers would face a positive marginal tax rate (see Brett and Weymark [2008c]). However, with learning-by-doing, it is not necessarily the case that, in the absence of taxation, type 2 consumers would choose to earn a higher income than type 1 consumers in the first period. This is because the lifetime marginal disutility that a type 1 consumer incurs from first-period labour may be less than that incurred by a type 2 consumer. (The reasoning is similar to that for the case when the government can commit, as discussed in Section 3.) Therefore, with learning-by-doing \( MTR_1^1 \) and \( MTR_2^1 \) cannot be signed because it is not clear whether each type’s labour supply is being distorted upwards or downwards to earn \( y_1 \). Furthermore, a marginal increase in \( y_1 \) will increase type 2’s first-period labour supply, which increases their second-period wage and thereby relaxes the incentive-compatibility constraint. An increase in \( w_2^2 \) relaxes the incentive-compatibility constraint because, in the second period, a type 2 consumer works longer when revealing herself than when mimicking. Therefore, there is a higher utility payoff under the former from a wage increase, which reduces the incentive to mimic. This gives the government a motive to distort type 2’s first-period labour supply upwards to relax the incentive-compatibility constraint, which interestingly is the opposite of the case when there is complete separation. But since the consumers
are completely pooled in period 1, an increase in $y^1$ used to increase type 2’s labour supply will also increase type 1’s labour supply. Thus it can be determined only that, in aggregate, first-period labour will be distorted upwards, i.e., the weighted sum of the first-period marginal tax rates must be negative.

4.3 Partial Pooling in Period 1

If in period 1 the government chose to pool a proportion $\gamma \in (0,1)$ of the high-skill consumers with the low-skill consumers, the government’s behaviour in period 2 can be described as follows. Choose $\langle c^2_1, y^2_1 \rangle$, $\langle c^2_2, y^2_2 \rangle$ and $\langle c^2_{2P}, y^2_{2P} \rangle$ to maximise:

$$(1-\phi) \left[ u(c^2_1) - v \left( \frac{y^2_1}{w^2_1} \right) \right] + (1-\gamma) \phi \left[ u(c^2_2) - v \left( \frac{y^2_2}{w^2_2} \right) \right] + \gamma \phi \left[ u(c^2_{2P}) - v \left( \frac{y^2_{2P}}{w^2_{2P}} \right) \right]$$

(4.11)

subject to:

$$(1-\phi) \left[ y^2_1 - c^2_1 \right] + (1-\gamma) \phi \left[ y^2_2 - c^2_2 \right] + \gamma \phi \left[ y^2_{2P} - c^2_{2P} \right] \geq 0$$

(4.12)

$$u(c^2_{2P}) - v \left( \frac{y^2_{2P}}{w^2_{2P}} \right) \geq u(c^2_1) - v \left( \frac{y^2_1}{w^2_1} \right)$$

(4.13)

where $\langle c^2_{2P}, y^2_{2P} \rangle$ is the tax contract offered to those high-skill consumers who were pooled in period 1, and $w^2_{2P}$ is their second-period wage. Equation (4.11) is the second-period social welfare function, (4.12) is the second-period budget constraint, and (4.13) is an incentive-compatibility constraint for those high-skill consumers who were pooled in the first period.\(^{18}\) Since $w^2_1 = w^2_1 \left( \frac{y^1_P}{w^1_1} \right)$, $w^2_2 = w^2_2 \left( \frac{y^1_1}{w^2_2} \right)$ and $w^2_{2P} = w^2_2 \left( \frac{y^1_P}{w^2_{2P}} \right)$, the solution to the above programme yields the functions $c^2_1(\phi, \gamma, y^1_P, y^1_1, w^1_1, w^1_2)$, $y^2_1(\phi, \gamma, y^1_P, y^1_1, w^1_1, w^1_2)$, $c^2_2(\phi, \gamma, y^1_P, y^1_2, w^1_1, w^1_2)$, $y^2_2(\phi, \gamma, y^1_P, y^1_2, w^1_1, w^1_2)$, $c^2_{2P}(\phi, \gamma, y^1_P, y^1_2, w^1_1, w^1_2)$ and $y^2_{2P}(\phi, \gamma, y^1_P, y^1_2, w^1_1, w^1_2)$, where $y^1_P$ denotes the pre-tax income earned by the low-skill and pooled high-skill consumers in period 1. Substituting these functions into (4.11) yields the value function $W^2(\phi, \gamma, y^1_P, y^1_1, w^1_1, w^1_2)$.

If there is partial pooling in period 1, the consumers and the government know that the government will solve programme (4.11) – (4.13) in period 2. The government in

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\(^{18}\)In period 2 the government can identify those high-skill consumers who were separated in the first period. It therefore does not face an incentive-compatibility constraint for those consumers.
period 1 therefore chooses \(\langle c_1^{P}, y_1^{P}\rangle\) and \(\langle c_1^{2}, y_2^{2}\rangle\) to maximise:

\[
(1 - \phi) \left[ u(c_1^{P}) - v \left( \frac{y_1^{P}}{w_1^{P}} \right) \right] + (1 - \gamma)\phi \left[ u(c_2^{2}) - v \left( \frac{y_2^{2}}{w_2^{2}} \right) \right] + \gamma\phi \left[ u(c_1^{P}) - v \left( \frac{y_1^{P}}{w_1^{P}} \right) \right] + W^2(\cdot)
\]

subject to:

\[
(1 - \phi + \gamma\phi) \left[ y_1^{P} - c_1^{P} \right] + (1 - \gamma)\phi \left[ y_2^{2} - c_2^{2} \right] \geq 0
\]

\[
u(c_2^{2}) - v \left( \frac{y_2^{2}}{w_2^{2}} \right) + u(c_2^{2}(\cdot)) - v \left( \frac{y_2^{2}(\cdot)}{w_2^{2}} \right) \geq u(c_1^{P}) - v \left( \frac{y_1^{P}}{w_1^{P}} \right) + u(c_2^{2}(\cdot)) - v \left( \frac{y_2^{2}(\cdot)}{w_2^{2}} \right)
\]

where \(\langle c_1^{P}, y_1^{P}\rangle\) is the tax contract offered to the low-skill and pooled high-skill consumers in period 1. When choosing \(\langle c_1^{P}, y_1^{P}\rangle\) and \(\langle c_1^{2}, y_2^{2}\rangle\), the government considers the affect on social welfare in period 2. Its first-period objective function (4.14) therefore includes the second-period value function \(W^2(\cdot)\). Equation (4.15) is the first-period budget constraint, while (4.16) is an incentive-compatibility constraint that ensures willing partial separation of the high-skill consumers in the first period.

Let \(MTR_1^{t}\) denote the marginal tax rate in period \(t\) faced by those high-skilled consumers who are pooled in period 1. It is shown in the appendix that the solutions to programmes (4.11) – (4.13) and (4.14) – (4.16) together imply:

**Proposition 4** Optimal income taxation with learning-by-doing, when the government cannot commit to its second-period tax policy, and when there is partial pooling in the first period is characterised by: \(MTR_1^{1} \geq 0\), \(MTR_2^{1} > 0\), \(MTR_1^{2} \geq 0\), \(MTR_2^{2} > 0\), \(MTR_2^{3} = 0\), and \(MTR_2^{3} = 0\).

In period 2, the government can identify those high-skill consumers who were separated in period 1. Therefore, they are subjected to lump-sum taxation in period 2 and face a zero marginal tax rate. The government is then left with the high-skill consumers who were pooled and the low-skill consumers. The government subjects these consumers to incentive-compatible taxation in period 2 as in a static model, yielding the standard marginal tax rate results.

In period 1, the government imposes a positive marginal tax rate on those high-skill consumers who are separated. The reasoning is similar to that for the case when all high-skill consumers are separated. That is, since the separated high-skill consumers are
subjected to lump-sum taxation in period 2, their second-period utility is decreasing in their second-period wage. Furthermore, the utility a pooled high-skill consumer obtains in period 2 is increasing in the separated high-skill consumers’ second-period wage, since this allows more redistribution towards the low-skill consumers who the pooled high-skill consumers can mimic. Thus distorting the separated high-skill consumers’ first-period labour supply downwards decreases their second-period wage, but increases their second-period utility and decreases the second-period utility obtained by the pooled high-skill consumers. Both these effects make mimicking less attractive, and therefore the first-period incentive-compatibility constraint is relaxed.

The first-period marginal tax rates faced by the low-skill and pooled high-skill consumers cannot be signed. This is for the same reasons as to why there is ambiguity when there is complete pooling, but now there are some additional factors at work. As in the case of complete pooling, the government has a motive to distort the pooled high-skill consumers’ first-period labour supply upwards to relax the second-period incentive-compatibility constraint. However, a marginal increase in $y^1_p$ used to increase the pooled high-skill consumers’ first-period labour supply would also require an increase in the low-skill consumers’ first-period labour supply. Thus the second-period wages of both the pooled high-skill consumers and the low-skill consumers would increase. As it is not clear how increases in these wages affects the extent of redistribution undertaken in period 2, the effect on the first-period incentive-compatibility constraint is also unclear. This further serves to render $MTR^1_1$ and $MTR^1_{2P}$ ambiguous.

5 Concluding Comments

The ‘new dynamic public finance’ literature that extends Mirrlees [1971] to a dynamic setting has assumed that random productivity shocks determine future wages, and that the government can commit to its future tax policy. The assumption that the government can commit is perhaps a strong one, since the present government cannot commit future governments. For example, Auerbach [2006] cites a proposal made to resolve the U.S. Social Security system’s imbalance, which includes a tax increase to be made in 2045!
As Auerbach notes, such a proposal cannot be taken seriously.

Recent contributions by Berliant and Ledyard [2005], Apps and Rees [2006], and Brett and Weymark [2008c] have dropped the commitment assumption, but they assume that wages are fixed. By contrast, we have assumed that learning-by-doing determines future wages, and that the government may not be able to commit. Given that the sole source of heterogeneity in the Mirrlees framework is wage differentials, understanding how optimal marginal tax rates respond to changes in wages seems particularly relevant.

It has long been known that endogenous wages in static models make it optimal for the high-skill type to face a negative marginal tax rate (see Stiglitz [1982]). Recently, Simula [2007] and Brett and Weymark [2008b] have derived a number of comparative static results for exogenous changes in wages. However, the ‘no-distortion-at-the-top’ result remains intact. Our analysis shows that in a dynamic model with wages determined by learning-by-doing, the ‘no-distortion-at-the-top’ result no longer applies, and a positive marginal tax rate on the high-skill type can be justified despite its depressing effect on labour supply and wages.

Given that learning-by-doing has been shown to imply that some of the optimal marginal tax rates cannot be signed, the question arises as to how useful the results are as a guide to designing real-world tax systems. Our preferred interpretation of the results as a guide to policy is that they show, contrary to intuition, that learning-by-doing does not necessarily make a case for lowering marginal tax rates. We also think that the economic intuition driving the results is more important and valuable to policymakers than the results themselves. This is related to what we think is a reasonable interpretation of the well-known Mirrlees/Stiglitz results that the high-skill type should face a zero marginal tax rate and the low-skill type should face a positive marginal tax rate. That is, the Mirrlees/Stiglitz results can simply be interpreted as showing that a redistributive tax system need not be characterised by an increasing pattern of marginal tax rates, as popular thinking and actual practice suggest. It does not seem reasonable to interpret the Mirrlees/Stiglitz results as an absolute call to design real-world tax systems in the manner that their results imply, since they are derived from highly-simplified models of the economy and extensions of their work (such as this paper)
have identified settings in which their results no longer apply.

To the best of our knowledge, this paper is the first to examine the implications of learning-by-doing for redistributive taxation. We have therefore maintained the most common assumptions made in the optimal nonlinear income tax literature, so that the effects of learning-by-doing can be isolated. In particular, we have assumed that the social welfare function is utilitarian and that only the high-skill type’s incentive-compatibility constraint may bind.\footnote{The related papers by Apps and Rees [2006] and Brett and Weymark [2008c] also make these assumptions.} In future work it might be interesting to examine the implications of using other social welfare functions, or allowing the low-skill type’s wage to become greater than that of the high-skill type as a result of learning-by-doing, which would create the possibility that the low-skill type’s incentive-compatibility constraint binds. These seem interesting avenues for future research.

\section{Appendix}

\textbf{Derivation of Equations (2.2) and (2.3)}

The Lagrangian corresponding to programme (2.1) can be written as:

\[ L = u(c^1_i) - v(l^1_i) + u(c^2_i) - v(l^2_i) \]

\[ + \alpha^1 \left[ w^1_i l^1_i - T^1(w^1_i l^1_i) - c^1_i \right] + \alpha^2 \left[ w^2_i l^2_i - T^2(w^2_i l^2_i) - c^2_i \right] \]

(A.1)

where $\alpha^1 \geq 0$ and $\alpha^2 \geq 0$ are Lagrange multipliers. The relevant first-order conditions can be written as:

\[ u'(c^1_i) - \alpha^1 = 0 \]

(A.2)

\[ -v'(l^1_i) + \alpha^1 w^1_i \left[ 1 - \frac{\partial T^1(.)}{\partial y^1_i} \right] + \alpha^2 l^2_i \frac{\partial w^1_i}{\partial l^1_i} \left[ 1 - \frac{\partial T^2(.)}{\partial y^2_i} \right] = 0 \]

(A.3)

\[ u'(c^2_i) - \alpha^2 = 0 \]

(A.4)

\[ -v'(l^2_i) + \alpha^2 w^2_i \left[ 1 - \frac{\partial T^2(.)}{\partial y^2_i} \right] = 0 \]

(A.5)
Straightforward manipulation of (A.4) and (A.5) yields equation (2.3). After substituting (2.3) into (A.3) and using (A.2) and (A.4) to eliminate the Lagrange multipliers, equation (A.3) can be manipulated to yield equation (2.2). □

Proof of Proposition 1
The relevant first-order conditions corresponding to programme (3.1) – (3.5) are:

\[(1 - \phi - \theta_2)u'(c_1^2) - \lambda^1(1 - \phi) = 0\]  *(A.6)*

\[-(1 - \phi)v'\left(\frac{y_1}{w_1}\right)\frac{1}{w_1} + (1 - \phi)v'\left(\frac{y_1^2}{w_1^2}\right)\frac{y_1}{w_1^2 w_1^1} \frac{\partial w_1^2(\cdot)}{\partial l_1^1} + \lambda^1(1 - \phi)\]

\[+ \theta_2 v'\left(\frac{y_1}{w_2}\right)\frac{1}{w_2} - \theta_2 v'\left(\frac{y_1^2}{w_2^2}\right)\frac{y_1^2}{w_2^2 w_2^1} \frac{\partial w_2^2(\cdot)}{\partial l_1^2} = 0\]  *(A.7)*

\[(\phi + \theta_2)u'(c_1^2) - \lambda^1 \phi = 0\]  *(A.8)*

\[-(\phi + \theta_2)v'\left(\frac{y_1}{w_1}\right)\frac{1}{w_1} + (\phi + \theta_2)v'\left(\frac{y_1^2}{w_1^2}\right)\frac{y_1^2}{w_1^2 w_1^1} \frac{\partial w_1^2(\cdot)}{\partial l_1^1} + \lambda^1 \phi = 0\]  *(A.9)*

\[(1 - \phi - \theta_2)u'(c_1^2) - \lambda^2(1 - \phi) = 0\]  *(A.10)*

\[-(1 - \phi)v'\left(\frac{y_1^2}{w_1^2}\right)\frac{1}{w_1^2} + \lambda^2(1 - \phi) + \theta_2 v'\left(\frac{y_1^2}{w_2^2}\right)\frac{1}{w_2^2} = 0\]  *(A.11)*

\[(\phi + \theta_2)u'(c_2^2) - \lambda^2 \phi = 0\]  *(A.12)*

\[-(\phi + \theta_2)v'\left(\frac{y_2}{w_2}\right)\frac{1}{w_2} + \lambda^2 \phi = 0\]  *(A.13)*

where \(\lambda^1 \geq 0\) is the multiplier on the first-period budget constraint (3.2), \(\lambda^2 \geq 0\) is the multiplier on the second-period budget constraint (3.3), \(\theta_2 \geq 0\) is the multiplier on type 2's incentive-compatibility constraint (3.5), and \(l_2^1 = y_1^1 / w_1^1\).

Dividing (A.13) by (A.12) and rearranging yields:

\[\frac{v'(l_2^2)}{u'(c_2^2)w_2^2} = 1\]  *(A.14)*

which using (2.3) establishes that \(MTR_2^2 = 0\). Similarly, dividing (A.9) by (A.8) and rearranging yields:

\[\frac{v'(l_2^1)}{u'(c_2^1)w_2^1} - \frac{v'(l_2^1)l_2^1}{u'(c_2^1)w_2^1 w_2^2} \frac{\partial w_2^2(\cdot)}{\partial l_1^2} = 1\]  *(A.15)*
which using (2.2) establishes that $MTR_1^1 = 0$.

Using (A.10) and (A.11) we obtain:

\[
(1 - \phi - \theta_2)u'(c_1^2) = (1 - \phi)v'\left(\frac{y_1^2}{w_1^2}\right) \frac{1}{w_1^2} - \theta_2 v'\left(\frac{y_1^2}{w_2^2}\right) \frac{1}{w_2^2}
\]  

(A.16)

Because $\bar{w}_2^2 > w_1^2$ and $v(\cdot)$ is strictly convex:

\[
(1 - \phi)v'\left(\frac{y_1^2}{w_1^2}\right) \frac{1}{w_1^2} - \theta_2 v'\left(\frac{y_1^2}{w_2^2}\right) \frac{1}{w_2^2} > (1 - \phi)v'\left(\frac{y_1^2}{w_1^2}\right) \frac{1}{w_1^2} - \theta_2 v'\left(\frac{y_1^2}{w_1^2}\right) \frac{1}{w_1^2}
\]  

(A.17)

Therefore, (A.16) and (A.17) imply that:

\[
(1 - \phi - \theta_2)u'(c_1^2) > (1 - \phi - \theta_2)v'\left(\frac{y_1^2}{w_1^2}\right) \frac{1}{w_1^2}
\]  

(A.18)

Using (A.6) and (A.8) it follows that $1 - \phi - \theta_2 > 0$. Hence (A.18) can be rearranged to yield:

\[
1 > \frac{v'(l_1^2)}{u'(c_1^2)w_1^2}
\]  

(A.19)

which using (2.3) establishes that $MTR_1^2 > 0$.

To show that $MTR_1^1$ is ambiguous, use (A.6) and (A.7) to obtain:

\[
(1 - \phi - \theta_2)u'(c_1^1) = (1 - \phi) \left[ v'\left(\frac{y_1^1}{w_1^1}\right) \frac{1}{w_1^1} - v'\left(\frac{y_1^2}{w_1^2}\right) \frac{y_1^2}{w_1^2 w_2^1 l_1^1} \frac{\partial w_2^2(\cdot)}{\partial l_1^1} \right]
\]

\[ - \theta_2 \left[ v'\left(\frac{y_1^1}{w_2^2}\right) \frac{1}{w_2^2} - v'\left(\frac{y_1^2}{w_2^2}\right) \frac{y_1^2}{w_2^2 w_1^1 l_1^2} \frac{\partial w_2^2(\cdot)}{\partial l_1^2} \right]
\]  

(A.20)

Since:

\[
v'\left(\frac{y_1^1}{w_1^1}\right) \frac{1}{w_1^1} - v'\left(\frac{y_1^2}{w_1^2}\right) \frac{y_1^2}{w_1^2 w_2^1 w_1^1} \frac{\partial w_2^2(\cdot)}{\partial l_1^1} \geq v'\left(\frac{y_1^1}{w_1^2}\right) \frac{1}{w_1^2} - v'\left(\frac{y_1^2}{w_1^2}\right) \frac{y_1^2}{w_1^2 w_2^1 w_1^1} \frac{\partial w_2^2(\cdot)}{\partial l_1^2}
\]

(A.21)

we obtain:

\[
(1 - \phi - \theta_2)u'(c_1^1) \geq (1 - \phi - \theta_2) \left[ v'\left(\frac{y_1^1}{w_1^1}\right) \frac{1}{w_1^1} - v'\left(\frac{y_1^2}{w_1^2}\right) \frac{y_1^2}{w_1^2 w_2^1 w_1^1} \frac{\partial w_2^2(\cdot)}{\partial l_1^1} \right]
\]  

(A.22)
Therefore:

$$1 = \frac{v'(l_1^0)}{u'(c_1^1)w_1^1} - \frac{v'(l_2^0)l_1^2}{u'(c_1^1)w_1^1w_2^2} \frac{\partial w_1^2(\cdot)}{\partial l_1^1}$$

(A.23)

which using (2.2) establishes that $MTR_1^1 \geq 0$. ■

**Proof of Proposition 2**

The first-order conditions corresponding to programme (4.1) – (4.2) are:

$$(1 - \phi)u'(c_1^2) - \lambda^2(1 - \phi) = 0$$

(A.24)

$$- (1 - \phi)v' \left( \frac{y_1^2}{w_1^2} \right) \frac{1}{w_1^2} + \lambda^2(1 - \phi) = 0$$

(A.25)

$$\phi u'(c_2^2) - \lambda^2 \phi = 0$$

(A.26)

$$- \phi v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} + \lambda^2 \phi = 0$$

(A.27)

$$(1 - \phi) \left[ y_1^2 - c_1^2 \right] + \phi \left[ y_2^2 - c_2^2 \right] = 0$$

(A.28)

where $\lambda^2 \geq 0$ is the multiplier on the second-period budget constraint (4.2). Dividing (A.25) by (A.24) and rearranging yields:

$$\frac{v'(l_1^2)}{u'(c_1^2)w_1^2} = 1$$

(A.29)

while dividing (A.27) by (A.26) and rearranging yields:

$$\frac{v'(l_2^2)}{u'(c_2^2)w_2^2} = 1$$

(A.30)

which using (2.3) establish that $MTR_1^2 = 0$ and $MTR_2^2 = 0$.

The relevant first-order conditions corresponding to programme (4.3) – (4.5) can be written as:

$$(1 - \phi - \theta_2^1)u'(c_1^1) - \lambda^1(1 - \phi) = 0$$

(A.31)

$$- (1 - \phi)v' \left( \frac{y_1^1}{w_1^1} \right) \frac{1}{w_1^1} + \frac{\partial W^2(\cdot)}{\partial y_1^1} + \lambda^1(1 - \phi) + \theta_2^1 v' \left( \frac{y_1^2}{w_2^2} \right) \frac{1}{w_2^2} - \theta_2^1 v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_1^2}{w_2^2w_1^2} \frac{\partial w_2^2(\cdot)}{\partial l_1^2}$$
\( + \theta_2 \left[ u'(c_2^2) \frac{\partial c_2^2}{\partial y_1^1} - v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} \frac{\partial y_2^2}{\partial y_1^1} \right] \) = 0
\[ \text{(A.32)} \]

\((\phi + \theta_2) u'(c_2^1) - \lambda^1 \phi = 0 \]
\[ \text{(A.33)} \]

\(- \phi v' \left( \frac{y_1^2}{w_2^2} \right) \frac{1}{w_2^2} + \frac{\partial W^2(\cdot)}{\partial y_2^1} + \lambda^1 \phi - \theta_2 v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2 w_2^2 w_2^1} \frac{\partial w_2^2(\cdot)}{\partial l_2^1} + \theta_2 \left[ u'(c_2^2) \frac{\partial c_2^2}{\partial y_2^2} - v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} \frac{\partial y_2^2}{\partial y_2^2} \right] = 0 \]
\[ \text{(A.34)} \]

where \( \lambda^1 \geq 0 \) is the multiplier on the first-period budget constraint (4.4), and \( \theta_2 \geq 0 \) is the multiplier on the incentive-compatibility constraint (4.5). To derive expressions for \( \frac{\partial W^2(\cdot)}{\partial y_1^1} \) and \( \frac{\partial W^2(\cdot)}{\partial y_2^1} \), note that the Lagrangian corresponding to programme (4.1) – (4.2) can be written as:

\[ L = (1 - \phi) \left[ u(c_1^2) - v \left( \frac{y_1^2}{w_1^2} \right) \right] + \phi \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) \right] + \lambda^2 \left[ (1 - \phi) [y_1^2 - c_1^2] + \phi [y_2^2 - c_2^2] \right] \]
\[ \text{(A.35)} \]

By the Envelope Theorem:

\[ \frac{\partial W^2(\cdot)}{\partial y_1^1} = \frac{\partial L(\cdot)}{\partial y_1^1} = (1 - \phi) v' \left( \frac{y_1^2}{w_1^2} \right) \frac{y_1^2}{w_1^2 w_1^2 w_1^1} \frac{\partial w_1^2(\cdot)}{\partial l_1^1} \]
\[ \text{(A.36)} \]

\[ \frac{\partial W^2(\cdot)}{\partial y_2^2} = \frac{\partial L(\cdot)}{\partial y_2^2} = \phi v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2 w_2^2 w_2^1} \frac{\partial w_2^2(\cdot)}{\partial l_2^1} \]
\[ \text{(A.37)} \]

Substituting (A.37) into (A.34) and combining the result with (A.33) yields:

\[ (\phi + \theta_2) v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} - (\phi + \theta_2) v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2 w_2^2 w_2^1} \frac{\partial w_2^2(\cdot)}{\partial l_2^1} = (\phi + \theta_2) u'(c_2^1) + \theta_2 \left[ u'(c_2^2) \frac{\partial c_2^2}{\partial y_1^1} - v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} \frac{\partial y_2^2}{\partial y_2^2} \right] \]
\[ \text{(A.38)} \]

Dividing both sides of (A.38) by \((\phi + \theta_2) u'(c_2^1)\) and rearranging yields:

\[ 1 - \frac{v'(l_1^2)}{u'(c_2^1) w_2^2} + \frac{v'(l_1^2) l_2^2}{u'(c_2^1) w_2^2 w_2^1} \frac{\partial w_2^2(\cdot)}{\partial l_2^1} = - \frac{\theta_2}{(\phi + \theta_2) u'(c_2^1) \frac{\partial c_2^2}{\partial y_2^1} - v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} \frac{\partial y_2^2}{\partial y_2^1} \right] \]
\[ 23 \]
where

\[ A \]

establishes that

\[ \begin{align*}
\frac{\partial c_2^2(\cdot)}{\partial y_2^1} &= \frac{-\theta_2^1 u'(c_2^2) v''(\frac{\hat{c}_1}{w_2})}{|A|} \left[ \frac{1}{w_1^2 w_1^2} \left[ v'' \left( \frac{\hat{c}_1}{w_2} \right) \frac{\hat{c}_1}{w_2} + v' \left( \frac{\hat{c}_1}{w_2} \right) \right] - \frac{\phi^3}{w_1^4 w_2^4} \frac{\partial u_2^2(\cdot)}{\partial l_2} \right] > 0 \tag{A.41} \\
\frac{\partial y_2^2(\cdot)}{\partial y_2^1} &= \frac{1 - \phi}{|A|} \left[ u''(c_2^2) v''(\frac{\hat{c}_1}{w_2}) \frac{1}{w_1^2 w_1^2} \left[ v'' \left( \frac{\hat{c}_1}{w_2} \right) \frac{\hat{c}_1}{w_2} + v' \left( \frac{\hat{c}_1}{w_2} \right) \right] \frac{\phi^3}{w_1^4 w_2^4} \frac{\partial u_2^2(\cdot)}{\partial l_2} \right] > 0 \tag{A.42} \\
\frac{\partial c_1^2(\cdot)}{\partial y_2^1} &= \frac{-\theta_2^1 u'(c_2^2) v''(\frac{\hat{c}_1}{w_2})}{|A|} \left[ \frac{1}{w_1^2 w_1^2} \left[ v'' \left( \frac{\hat{c}_1}{w_2} \right) \frac{\hat{c}_1}{w_2} + v' \left( \frac{\hat{c}_1}{w_2} \right) \right] - \frac{\phi^3}{w_1^4 w_2^4} \frac{\partial u_2^2(\cdot)}{\partial l_2} \right] > 0 \tag{A.43} \\
\frac{\partial y_2^2(\cdot)}{\partial y_2^1} &= \frac{(1 - \phi) u''(c_2^2) v''(\frac{\hat{c}_1}{w_2})}{|A|} \left[ \frac{1}{w_1^2 w_1^2} \left[ v'' \left( \frac{\hat{c}_1}{w_2} \right) \frac{\hat{c}_1}{w_2} + v' \left( \frac{\hat{c}_1}{w_2} \right) \right] \frac{\phi^3}{w_1^4 w_2^4} \frac{\partial u_2^2(\cdot)}{\partial l_2} \right] < 0 \tag{A.44}
\end{align*} \]

where A is the Hessian associated with (A.24) – (A.28):

\[
A = \begin{bmatrix}
(1 - \phi) u''(c_2^2) & 0 & 0 & 0 & -(1 - \phi) \\
0 & -(1 - \phi) v''(\frac{\hat{c}_1}{w_2}) & \frac{1}{w_1^2 w_1^2} & 0 & 1 - \phi \\
0 & 0 & \phi u''(c_2^2) & 0 & -\phi \\
0 & 0 & 0 & -\phi v''(\frac{\hat{c}_1}{w_2}) & \phi \\
-(1 - \phi) & 1 - \phi & -\phi & \phi & 0
\end{bmatrix} \tag{A.45}
\]
and the determinant of $A$ is given by:

$$
|A| = [1 - \phi(2 - \phi)] u''(c_2^2) v'' \left( \frac{y_2^2}{w_2^2} \right) \frac{\phi^2}{w_1^2 w_2^2} (1 - \phi) \left( u''(c_1^2) - v'' \left( \frac{y_1^2}{w_1^2} \right) \frac{1}{w_1^2 w_2^2} \right)
$$

$$
+ (1 - \phi)^2 u''(c_1^2) v'' \left( \frac{y_1^2}{w_1^2} \right) \frac{\phi^3}{w_1^2 w_1^2} \left( u''(c_2^2) - v'' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2 w_2^2} \right) > 0 \quad (A.46)
$$

Therefore, using (A.41) and (A.42) it can be shown that $\frac{\partial c_2^2(-)}{\partial y_2^2} - \frac{\partial y_2^2(-)}{\partial y_2^2} < 0$, which along with (A.43) and (A.44) establishes that $MTR_2^1 > 0$.

To show that $MTR_1^1$ is ambiguous, use (A.31), (A.32) and (A.36) to obtain:

$$
(1 - \phi - \theta^1) u'(c_1^1) = (1 - \phi) \left[ u' \left( \frac{y_1^1}{w_1^1} \right) \frac{1}{w_1^1} - u' \left( \frac{y_1^2}{w_1^2} \right) \frac{y_1^2}{w_1^2} \frac{\partial w_2^2(-)}{\partial l_1^1} \right]
$$

$$
- \theta^1 \left[ u' \left( \frac{y_1^1}{w_2^1} \right) \frac{1}{w_2^1} - u' \left( \frac{y_1^2}{w_2^2} \right) \frac{y_1^2}{w_2^2} \frac{\partial w_2^2(-)}{\partial l_1^1} \right]
$$

$$
- \theta^1 \left[ u'(c_2^2) \frac{\partial c_2^2(-)}{\partial y_1^1} - v' \left( \frac{y_2^2}{w_2^2} \right) \frac{\partial y_2^2(-)}{\partial y_1^1} \right] + \theta^1 \left[ u'(c_1^1) \frac{\partial c_1^2(-)}{\partial y_1^1} - v' \left( \frac{y_1^2}{w_1^2} \right) \frac{1}{w_1^2} \frac{\partial y_2^2(-)}{\partial y_1^1} \right] \quad (A.47)
$$

Using (2.2), (A.26) and (A.27), equation (A.47) can be manipulated to produce:

$$
MTR_1^1 = \frac{\theta^1}{1 - \phi} \left[ 1 - \frac{1}{u'(c_1^1)} \left[ u' \left( \frac{y_1^1}{w_1^1} \right) \frac{1}{w_1^1} - u' \left( \frac{y_1^2}{w_2^2} \right) \frac{y_1^2}{w_2^2} \frac{\partial w_2^2(-)}{\partial l_1^1} \right] \right]
$$

$$
- \frac{\theta^1 u'(c_2^2)}{(1 - \phi) u'(c_1^1)} \left[ \frac{\partial c_2^2(-)}{\partial y_1^1} - \frac{\partial y_2^2(-)}{\partial y_1^1} \right] + \frac{\theta^1}{(1 - \phi) u'(c_1^1)} \left[ u'(c_1^1) \frac{\partial c_1^2(-)}{\partial y_1^1} - v' \left( \frac{y_1^2}{w_1^2} \right) \frac{1}{w_1^2} \frac{\partial y_2^2(-)}{\partial y_1^1} \right] \quad (A.48)
$$

The sign of the first term on the right-hand side of (A.48) is ambiguous for the same reasons as to why $MTR_1^1$ is ambiguous when the government can commit. Using techniques similar to those used to sign $\frac{\partial c_2^2(-)}{\partial y_2^2} - \frac{\partial y_2^2(-)}{\partial y_2^2}$, it can be shown that $\frac{\partial c_2^2(-)}{\partial y_1^1} - \frac{\partial y_2^2(-)}{\partial y_1^1} > 0$ and therefore the second term on the right-hand side of (A.48) is negative. It can also be shown that $\frac{\partial c_2^2(-)}{\partial y_1^1} > 0$, $\frac{\partial y_2^2(-)}{\partial y_1^1} > 0$ and $\frac{\partial c_2^2(-)}{\partial y_1^1} - \frac{\partial y_2^2(-)}{\partial y_1^1} < 0$. However, since $u'(c_1^1) > v' \left( \frac{y_1^2}{w_1^2} \right) \frac{1}{w_1^2}$, the last term in (A.48) cannot be signed. 

---

\textsuperscript{20} Using (A.24) and (A.25) we obtain $u'(c_1^1) = v' \left( \frac{y_1^1}{w_1^1} \right) \frac{1}{w_1^1} > v' \left( \frac{y_1^2}{w_2^2} \right) \frac{1}{w_2^2}$. 

25
Proof of Proposition 3

When there is complete pooling in period 1, the second-period optimal tax problem is identical to that in a static model. We therefore omit the proof of the results that $MTR_{11} > 0$ and $MTR_{22} = 0$.

The relevant first-order conditions corresponding to programme (4.9) – (4.10) can be written as:

\[ u'(c_1) - \lambda_1 = 0 \]  
\[ -(1 - \phi)v' \left( \frac{y_1}{w_1} \right) \frac{1}{w_1} - \phi v' \left( \frac{y_1}{w_2} \right) \frac{1}{w_2} + \frac{\partial W^2(\cdot)}{\partial y^1} + \lambda_1 = 0 \]  
(A.49)

where $\lambda_1 \geq 0$ is the multiplier on the first-period budget constraint (4.10). To derive an expression for $\frac{\partial W^2(\cdot)}{\partial y^1}$, note that the Lagrangian corresponding to programme (4.6) – (4.8) can be written as:

\[ L = (1 - \phi) \left[ u(c_1^2) - v \left( \frac{y_1^2}{w_1^2} \right) \right] + \phi \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) \right] \]

\[ + \lambda_2 \left[ (1 - \phi) \left[ y_1^2 - c_1^2 \right] + \phi \left[ y_2^2 - c_2^2 \right] \right] + \theta_2^2 \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) - u(c_1^2) + v \left( \frac{y_1^2}{w_1^2} \right) \right] \]  
(A.50)

where $\lambda_2 \geq 0$ is the multiplier on the second-period budget constraint (4.7), and $\theta_2^2 \geq 0$ is the multiplier on the incentive-compatibility constraint (4.8). By the Envelope Theorem:

\[ \frac{\partial W^2(\cdot)}{\partial y^1} = \frac{\partial L(\cdot)}{\partial y^1} = (1 - \phi)v' \left( \frac{y_1^2}{w_1^2} \right) \frac{y_1^2}{w_1^2 w_1^1} \frac{\partial w_1^1(\cdot)}{\partial l_1^1} + \phi v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2 w_2^1} \frac{\partial w_2^1(\cdot)}{\partial l_1^1} \]

\[ + \theta_2^2 v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2 w_2^1} \frac{\partial w_2^1(\cdot)}{\partial l_2^1} - \theta_2^2 v' \left( \frac{y_1^2}{w_1^2} \right) \frac{y_1^2}{w_1^2 w_1^1} \frac{\partial w_1^1(\cdot)}{\partial l_2^1} \]  
(A.51)

Using equations (2.2), (A.49), (A.50) and (A.52) we obtain:

\[ (1 - \phi)MTR_{11} + \phi MTR_{22} = \frac{-\theta_2^2}{w'(c_1)w_1^2 w_2^2 w_1^1} \frac{\partial w_1^1(\cdot)}{\partial l_2^1} \left[ v' \left( \frac{y_2^2}{w_2^2} \right) y_2^2 - v' \left( \frac{y_1^2}{w_1^2} \right) y_1^2 \right] \]  
(A.53)

Since $v(\cdot)$ is strictly convex and $y_2^2 > y_1^2$, the term in square brackets in (A.53) is

\[ \text{Since } v(\cdot) \text{ is strictly convex and } y_2^2 > y_1^2, \text{ the term in square brackets in (A.53) is} \]

\[ 21 \text{Since the second-period optimal tax problem is identical to that in a static model, } y_2^2 > y_1^2 \text{ follows from the well-known result that it is optimal for the high-skill type to earn a higher pre-tax income than the low-skill type in the standard model.} \]
positive, which implies that $(1 - \phi)MTR^1_1 + \phi MTR^1_2 < 0$. Therefore $MTR^1_1 < 0$ and/or $MTR^1_2 < 0$, but $MTR^1_1$ and $MTR^1_2$ cannot be signed.

**Proof of Proposition 4**

The first-order conditions corresponding to programme (4.11) – (4.13) are:

$$(1 - \phi - \theta^2_{2P})u'(c^2_1) - \lambda^2(1 - \phi) = 0 \quad (A.54)$$

$$-(1 - \phi)u'\left(\frac{y^2_1}{w^2_1}\right) \frac{1}{w^2_1} + \lambda^2(1 - \phi) + \theta^2_{2P}v'\left(\frac{y^2_1}{w^2_{2P}}\right) \frac{1}{w^2_{2P}} = 0 \quad (A.55)$$

$$(\gamma \phi + \theta^2_{2P})u'(c^2_{2P}) - \lambda^2 \gamma \phi = 0 \quad (A.56)$$

$$-(\gamma \phi + \theta^2_{2P})v'\left(\frac{y^2_{2P}}{w^2_{2P}}\right) \frac{1}{w^2_{2P}} + \lambda^2 \gamma \phi = 0 \quad (A.57)$$

$$(1 - \gamma)\phi u'(c^2_2) - \lambda^2(1 - \gamma)\phi = 0 \quad (A.58)$$

$$-(1 - \gamma)\phi v'\left(\frac{y^2_2}{w^2_2}\right) \frac{1}{w^2_2} + \lambda^2(1 - \gamma)\phi = 0 \quad (A.59)$$

$$(1 - \phi)\left[y^2_1 - c^2_1\right] + (1 - \gamma)\phi \left[y^2_2 - c^2_2\right] + \gamma \phi \left[y^2_{2P} - c^2_{2P}\right] = 0 \quad (A.60)$$

$$u(c^2_{2P}) - v\left(H^2_{2P}\right) - u(c^1_1) + v\left(H^2_{2P}\right) = 0 \quad (A.61)$$

where $\lambda^2 \geq 0$ is the multiplier on the second-period budget constraint (4.12), and $\theta^2_{2P} \geq 0$ is the multiplier on the incentive-compatibility constraint (4.13). Dividing (A.57) by (A.56) and rearranging yields:

$$\frac{v'(l^2_{2P})}{u'(c^2_{2P})w^2_{2P}} = 1 \quad (A.62)$$

while dividing (A.59) by (A.58) and rearranging yields:

$$\frac{v'(l^2_{2})}{u'(c^2_{2})w^2_{2}} = 1 \quad (A.63)$$

which using (2.3) establish that $MTR^2_{2P} = 0$ and $MTR^2_{2} = 0$.

Using (A.54) and (A.55) we obtain:

$$(1 - \phi - \theta^2_{2P})u'(c^2_1) = (1 - \phi)v'\left(\frac{y^2_1}{w^2_1}\right) \frac{1}{w^2_1} - \theta^2_{2P}v'\left(\frac{y^2_1}{w^2_{2P}}\right) \frac{1}{w^2_{2P}} \quad (A.64)$$
Because $w^2_{2p} > w^2_1$ and $v(\cdot)$ is strictly convex:

$$(1 - \phi)v' \left( \frac{y^2_1}{w^2_1} \right) \frac{1}{w^2_1} - \theta^2_{2p} v' \left( \frac{y^2_1}{w^2_{2p}} \right) \frac{1}{w^2_{2p}} > (1 - \phi)v' \left( \frac{y^2_1}{w^2_1} \right) \frac{1}{w^2_1} - \theta^2_{2p} v' \left( \frac{y^2_1}{w^2_1} \right) \frac{1}{w^2_1} \quad \text{(A.65)}$$

Therefore, (A.64) and (A.65) imply that:

$$(1 - \phi - \theta^2_{2p}) u'(c^2_1) > (1 - \phi - \theta^2_{2p}) v' \left( \frac{y^2_1}{w^2_1} \right) \frac{1}{w^2_1} \quad \text{(A.66)}$$

Using (A.54) and (A.56) it follows that $1 - \phi - \theta^2_{2p} > 0$. Hence (A.66) can be rearranged to yield:

$$1 > \frac{v'(l^2_1)}{u'(c^1_1) w^2_1} \quad \text{(A.67)}$$

which using (2.3) establishes that $MTR^1_1 > 0$.

The relevant first-order conditions corresponding to programme (4.14) – (4.16) can be written as:

$$(1 - \phi + \gamma \phi - \theta^1_2) u'(c^1_2) - \lambda^1 (1 - \phi + \gamma \phi) = 0 \quad \text{(A.68)}$$

$$(1 - \phi)v' \left( \frac{y^1_2}{w^1_2} \right) \frac{1}{w^1_2} + \gamma \phi v' \left( \frac{y^1_2}{w^1_2} \right) \frac{1}{w^1_2} + \lambda^1 (1 - \phi + \gamma \phi) + \theta^1_2 v' \left( \frac{y^1_2}{w^1_2} \right) \frac{1}{w^1_2} - \theta^1_2 v' \left( \frac{y^2_1}{w^1_1} \right) \frac{1}{w^1_1} - \frac{\partial W^2(\cdot)}{\partial y^1_1} + \theta^1_2 \left[ u'(c^2_2) \frac{\partial c^2_2(\cdot)}{\partial y^1_1} - v' \left( \frac{y^2_1}{w^1_2} \right) \frac{1}{w^1_2} \frac{\partial y^2_1(\cdot)}{\partial y^1_1} \right] - \theta^1_2 \left[ u'(c^2_2) \frac{\partial c^2_2(\cdot)}{\partial y^1_1} - v' \left( \frac{y^2_1}{w^1_2} \right) \frac{1}{w^1_2} \frac{\partial y^2_1(\cdot)}{\partial y^1_1} \right] = 0 \quad \text{(A.69)}$$

$$[(1 - \gamma) \phi + \theta^1_2] u'(c^1_2) - \lambda^1 (1 - \gamma) \phi = 0 \quad \text{(A.70)}$$

$$(1 - \gamma) v' \left( \frac{y^1_2}{w^1_2} \right) \frac{1}{w^1_2} + \frac{\partial W^2(\cdot)}{\partial y^2_2} + \lambda^1 (1 - \gamma) \phi - \theta^1_2 v' \left( \frac{y^1_2}{w^1_2} \right) \frac{1}{w^1_2} + \theta^1_2 v' \left( \frac{y^2_1}{w^2_2} \right) \frac{1}{w^2_2} \frac{\partial y^2_1(\cdot)}{\partial y^2_2} + \theta^1_2 \left[ u'(c^2_2) \frac{\partial c^2_2(\cdot)}{\partial y^1_2} - v' \left( \frac{y^2_1}{w^2_2} \right) \frac{1}{w^2_2} \frac{\partial y^2_1(\cdot)}{\partial y^1_2} \right] - \theta^1_2 \left[ u'(c^2_2) \frac{\partial c^2_2(\cdot)}{\partial y^1_2} - v' \left( \frac{y^2_1}{w^2_2} \right) \frac{1}{w^2_2} \frac{\partial y^2_1(\cdot)}{\partial y^1_2} \right] = 0 \quad \text{(A.71)}$$

where $\lambda^1 \geq 0$ is the multiplier on the first-period budget constraint (4.15), and $\theta^1_2 \geq 0$ is the multiplier on the incentive-compatibility constraint (4.16). To derive expressions for $\partial W^2(\cdot)/\partial y^1_1$ and $\partial W^2(\cdot)/\partial y^1_2$, note that the Lagrangian corresponding to programme
(4.11) – (4.13) can be written as:

\[ L = (1 - \phi) \left[ u(c_1^2) - v \left( \frac{y_1^2}{w_1^2} \right) \right] + (1 - \gamma) \phi \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) \right] + \gamma \phi \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) \right] \]

\[ + \lambda^2 \left[ (1 - \phi) [y_1^2 - c_1^2] + (1 - \gamma) \phi [y_2^2 - c_2^2] + \gamma \phi [y_2^2 - c_2^2] \right] \]

\[ + \theta_2^2 \left[ u(c_2^2) - v \left( \frac{y_2^2}{w_2^2} \right) - u(c_1^2) + v \left( \frac{y_1^2}{w_1^2} \right) \right] \]

By the Envelope Theorem:

\[ \frac{\partial W^2(\cdot)}{\partial y_1^2} = \frac{\partial L(\cdot)}{\partial y_1^2} = (1 - \phi) v' \left( \frac{y_1^2}{w_1^2} \right) \frac{y_1^2}{w_1^2} \frac{\partial w_1^2(\cdot)}{\partial l_1^1} + \gamma \phi v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2} \frac{\partial w_2^2(\cdot)}{\partial l_1^2} \]

\[ + \frac{\theta_2^2}{w_2^2 w_2^2} \frac{\partial w_2^2(\cdot)}{\partial l_2^2} \left[ v' \left( \frac{y_2^2}{w_2^2} \right) y_2^2 - v' \left( \frac{y_1^2}{w_1^2} \right) y_1 \right] \]

\[ \frac{\partial W^2(\cdot)}{\partial y_2^1} = \frac{\partial L(\cdot)}{\partial y_2^1} = (1 - \gamma) \phi v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2} \frac{\partial w_2^2(\cdot)}{\partial l_2^2} \]

Substituting (A.74) into (A.71) and combining the result with (A.70) yields:

\[ \left[ (1 - \gamma) \phi + \theta_2^1 \right] v' \left( \frac{y_2^2}{w_2^2} \right) \frac{1}{w_2^2} - \left[ (1 - \gamma) \phi + \theta_2^1 \right] v' \left( \frac{y_2^2}{w_2^2} \right) \frac{y_2^2}{w_2^2} \frac{\partial w_2^2(\cdot)}{\partial l_2^2} \]

\[ = \left[ (1 - \gamma) \phi + \theta_2^1 \right] u'(c_2) \theta_2^1 u'(c_2) \left[ \frac{\partial c_2^2(\cdot)}{\partial y_2^1} - \frac{\partial y_2^2(\cdot)}{\partial y_2^1} \right] - \theta_2^1 u'(c_2^2) \left[ \frac{\partial c_2^2(\cdot)}{\partial y_2^1} - \frac{\partial y_2^2(\cdot)}{\partial y_2^1} \right] \]

where use has been made of (A.56) and (A.57), as well as (A.58) and (A.59). Equation (A.75) can be simplified to:

\[ \frac{v'(l_2^1)}{u'(c_2^1) w_2^1} - \frac{v'(l_2^1)}{u'(c_2^1) w_2^1} \frac{\partial w_2^2(\cdot)}{\partial l_2^1} = 1 \]

\[ = \frac{\theta_2^1 u'(c_2^1)}{(1 - \gamma) \phi + \theta_2^1} \left[ \frac{\partial c_2^2(\cdot)}{\partial y_2^1} - \frac{\partial y_2^2(\cdot)}{\partial y_2^1} \right] - \frac{\theta_2^1 u'(c_2^2)}{(1 - \gamma) \phi + \theta_2^1} \left[ \frac{\partial c_2^2(\cdot)}{\partial y_2^1} - \frac{\partial y_2^2(\cdot)}{\partial y_2^1} \right] \]

which using (2.2) and (A.70) reduces to:

\[ MTR_2^1 = \frac{-\theta_2^1 u'(c_2^1)}{\lambda^1(1 - \gamma) \phi} \left[ \frac{\partial c_2^2(\cdot)}{\partial y_2^1} - \frac{\partial y_2^2(\cdot)}{\partial y_2^1} \right] + \frac{\theta_2^1 u'(c_2^2)}{\lambda^1(1 - \gamma) \phi} \left[ \frac{\partial c_2^2(\cdot)}{\partial y_2^1} - \frac{\partial y_2^2(\cdot)}{\partial y_2^1} \right] \]
By applying the Implicit Function Theorem and Cramer’s Rule to (A.54) – (A.61), it can be shown that \( \frac{\partial \phi_2(z)}{\partial y_2} - \frac{\partial \phi_2(z)}{\partial y_1} < 0 \) and \( \frac{\partial \phi_2(z)}{\partial y_2} - \frac{\partial \phi_2(z)}{\partial y_1} > 0 \). Hence \( \text{MTR}_{R1} > 0 \).

Substituting (A.73) into (A.69) and combining the result with (A.68) yields:

\[
(1 - \phi + \gamma \phi - \theta_2^2) u'(c_p^*) = (1 - \phi) u'\left( \frac{y_1^2}{w_1^2} \right) - \frac{y_1^2}{w_1^2} \cdot \frac{1}{w_1^2} + \gamma \phi u'\left( \frac{y_1^2}{w_2^2} \right) - \frac{y_1^2}{w_2^2} \cdot \frac{1}{w_2^2} - (1 - \phi) u'\left( \frac{y_1^2}{w_2^2} \right) \cdot \frac{y_1^2}{w_2^2} \cdot \frac{1}{w_1^2} \cdot \frac{\partial w_1^2(z)}{\partial y_1} \\
- \gamma \phi u'\left( \frac{y_2^2}{w_2^2} \right) \cdot \frac{\partial w_2^2(z)}{\partial y_2} - \theta_2^2 u'\left( \frac{y_2^2}{w_2^2} \right) \cdot \frac{\partial w_2^2(z)}{\partial y_2} + \theta_2^1 u'\left( \frac{y_2^2}{w_2^2} \right) \cdot \frac{\partial w_2^2(z)}{\partial y_2} + \theta_2^1 u'\left( \frac{y_2^2}{w_2^2} \right) \cdot \frac{\partial w_2^2(z)}{\partial y_2} \\
- \theta_2^2 u'(c_p^*) \left[ \frac{\partial c_2^2(z)}{\partial y_1} - \frac{\partial y_2^2(z)}{\partial y_1} \right] + \theta_2^1 u'(c_p^*) \left[ \frac{\partial c_2^2(z)}{\partial y_1} - \frac{\partial y_2^2(z)}{\partial y_1} \right] (A.78)
\]

where use has been made of (A.56) and (A.57), as well as (A.58) and (A.59). Using (2.2), equation (A.78) can be simplified to:

\[
(1 - \phi) \text{MTR}_{R1} + (\gamma \phi - \theta_2) \text{MTR}_{R2P} = -\frac{\theta_2^2}{u'(c_p^*) \cdot \frac{\partial w_2^2(z)}{\partial y_2}} + \frac{\partial w_2^2(z)}{\partial y_2} \left[ u'\left( \frac{y_2^2}{w_2^2} \right) \cdot \frac{y_2^2}{w_2^2} - u'\left( \frac{y_1^2}{w_2^2} \right) \cdot \frac{y_1^2}{w_2^2} \right] \\
- \frac{\theta_2^1}{u'(c_p^*)} \left[ \frac{\partial c_2^2(z)}{\partial y_1} - \frac{\partial y_2^2(z)}{\partial y_1} \right] + \frac{\theta_2^1}{u'(c_p^*)} \left[ \frac{\partial c_2^2(z)}{\partial y_1} - \frac{\partial y_2^2(z)}{\partial y_1} \right] (A.79)
\]

The first term on the right-hand side of (A.79) is negative, as it is for the case of complete pooling (cf. equation (A.53)). In principle, one could use the Implicit Function Theorem and Cramer’s Rule to sign the last two terms on the right-hand side of (A.79), but this would require determining the comparative statics of a second-best nonlinear income tax problem, which are generally too complex to yield unambiguous results. Nevertheless, even if the last two terms in (A.79) could be signed, \( \text{MTR}_{R1} \) and \( \text{MTR}_{R2P} \) would remain ambiguous.

\[\text{References}\]

22Thus far, it has only been possible to derive the comparative static properties of optimal nonlinear income taxes for the case when preferences are quasi-linear. See Weymark [1987] and Brett and Weymark [2008a, 2008b] for the case when preferences are quasi-linear in labour, and Simula [2007] for the case when preferences are quasi-linear in consumption. It is not possible to impose quasi-linearity in our model, since it renders either the first-best or second-best taxation problems indeterminate.
References


