# A topologically-based classification of the 2x2 ordinal games

David Robinson, Department of Economics, Laurentian University
David Goforth, Department of Mathematics and Computer Science, Laurentian University

## Abstract

We describe a preference-based topology of the 2x2 ordinal games that provides a natural and systematic basis for classification. Games which are topologically close share economically significant features. In general, specific behavioural rules or solution concepts combine with the structural features of games to yield different partitions of the entire topological space of games. Economic features with clear topological foundations include symmetry, multiple equilibria, inferior equilibria, common interest, opposed interest, and constant rank-sum games. We provide a series of maps of subspaces induced by restricting the notion of neighbourhood in a natural way. We briefly compare the topologically based classification with typologies produced Rapoport and Guyer (1966), developed further in Rapoport, Guyer and Gordon, (1976) and Brams. (1994).

## Table of Contents



## Table of Figures

# A topologically-based classification of the 2x2 ordinal games

> "In every subject" "one looks for the topological and algebraic structures involved, since these structures form a unifying core for the most varied branches of mathematics".
>
> Weise and Noack, "Aspects of topology", Fundamentals of Mathematics: vol 2, Geometry Ch 16, 1986. English translation of Grundzhge der Mathematcik, second German edition 1967 and 1971. p593).

## 1 Introduction: Topologies and Typologies

Raopoprt and Guyer (1966), Rapoport, Guyer and Gordon (1976) and Brahms, (1994) have produced typologies of the 2x2 games. Historically, typologies belong to an early stage of science. In any scientific field typologies are replaced as soon as the deeper relationships are understood. Examples of typologies that have been replaced by systems based on deeper relationships abound. Members of the plant kingdom have been classified according to whether they are tall or short, for example. Bamboo then falls in the same group as the alder tree. Once it is clear that bamboo is much closer to wheat than to alder, we begin moving toward an approach based on how species may be derived from common ancestors. This "phyletic" approach demands a strict concept of what it means to be related.

A topology is in essence a specification of relationships among the members of a set. The first figures to be considered topologically were polyhedra, and we find in fact that the relationships among the 2x2 games can be understood by representing the 2x2 games either as the vertices or as the facets of a polyhedron. Connected sub-sets of games can be presented as smaller polyhedra.

Most introductions to topology begin by considering deformations of Euclidian surfaces like torii or Möebius strips. This approach treats topology in terms of "one-to-one bi-continuous mappings of sets of points in Euclidian space." It is an unnecessarily special approach according to Weise and Noack. One can instead generate a topological space from an arbitrary set of abstract elements, called “points,“ by imposing a topology on the set. Modern topology in fact takes no account of the individual nature of the elements, but merely of their mutual relationships.

In an approach based on points, however what is meant by the expression “neighbourhood of a point” must be defined axiomatically. Developing a satisfactory notion of "neighbour'" is the key to developing a topological treatment of the 2x2 games. Preferences provide enough structure to induce a topology on the 2x2 ordinal games that allows us to examine systematically the relationships among the 144 games[1].

In this paper we develop a complete description of the topology of the 2x2 games. In section 2 we introduce the swap operations that connect games. In section 3 we examine the graph of the network that represents the topological space of the 2x2 games. The section is intended to provide readers with a thorough understanding of the topology. Section 4 describes subspaces and

[1] The conventional view is that there are 78 2x2 games is correct but misleading, as we shall show.

regions that are of interest for economists. Section 5 examines the relationship between the Rapoport and Guyer typology, which is perhaps the best known classification scheme for the 2x2 games, and the underlying topology. Section 6 makes the usual kind of grandiose concluding statements. (This is one way to see if you are paying attention, David)

We will discuss the typology for the 2x2 games proposed by Rapoport and Guyer (1966), and revised later by Rapoport, Guyer and Gordon (1976) classification briefly when we have developed a description based on more fundamental relationships among of the 2x2 games. It will turn out that the features they use to produce their typology arise naturally from the topological structure, and that difficulties in their classification vanish in our approach

# 2 The swap operator and the neighbourhood.

## *2.1 What is a neighbour?*

Since games are characterised by the payoff function, similar games must have similar payoff functions. To define a neighbourhood, we need to characterise the smallest significant change in the payoff function.

Obviously a change affecting the payoffs of one player is smaller than a change affecting two players. A neighbouring game is therefore a game that differs only in that some small change has been made in the sequence of the four numbers that describe the ordinal payoffs for one player. At this point the economic structure of games becomes relevant. The nature of the preferences that are the basis of a player's decisions tells us that 1 is closer to 2 than it is to three. Similarly, 2 is closer to 3 than to 4. It follow that there are only three "smallest" changes for strict ordinal 2x2 games: [(1, 2) ⇔ (2, 1); (2, 3) ⇔ (3, 2); (3, 4) ⇔ (4, 3)]. Since the three swaps can be applied to the payoffs for either player, it follows that every game has exactly six nearest neighbours. The structure of preferences thus induces a topology on the set of games.

More formally, let Xij, (i∈{1..3}, j=i+1, X∈(R, C)) indicate the operation that changes the rank of the outcome originally ranked i by the player X to rank j and the rank of the outcome originally ranked j to rank i. When X=R we call this a "row swap". A Row Swap changes the ranking of two outcomes for the row player.

The row swap Rij is an operation on a game[2]: Rij(*g*) = *h*, where *g* and *h* are games. There is analogous column swap $C_{12}$. Row and column swaps convert games to their nearest neighbours. There are 6 order swaps satisfying j-I=1. Using these minimal swap operators, we define the neighbourhood of game *g* ∈{1..144}, N(*g*), as the set of games that can be reached by a single order swap.

N(*g*) = { Xij(*g*), | i∈{1..3}, j=i+1, X∈(R, C) }

"Neighbour of" is thus defined strictly in terms of the preferences of the players. The preference relation induces the topology.

## 2.2 Example: N(124), the neighbourhood of Game 124

Examining the neighbourhood of a specific game will make the operations more concrete and provide a sense of what the topological approach offers. Figure 1 illustrates the neighbourhood of game 124 using order graphs[3].

Game 124 is of interest in its own right. Like the Prisoner's Dilemma (PD), it has a single Pareto-dominated Nash equilibrium. Unlike the PD, only one player has a dominant strategy. The game is clearly closely related to the PD in important ways. It will generate social dilemmas as interesting as those generated by the PD. Furthermore, since game 124 is asymmetric, it is possible that it might characterize an even larger set of social situations that the PD does.

---

[2] Or alternatively, a mapping from the space of 2x2 games to itself.

[3] An order graph is a device we have developed for expositional purposes. Similar devices have been used by other researchers.

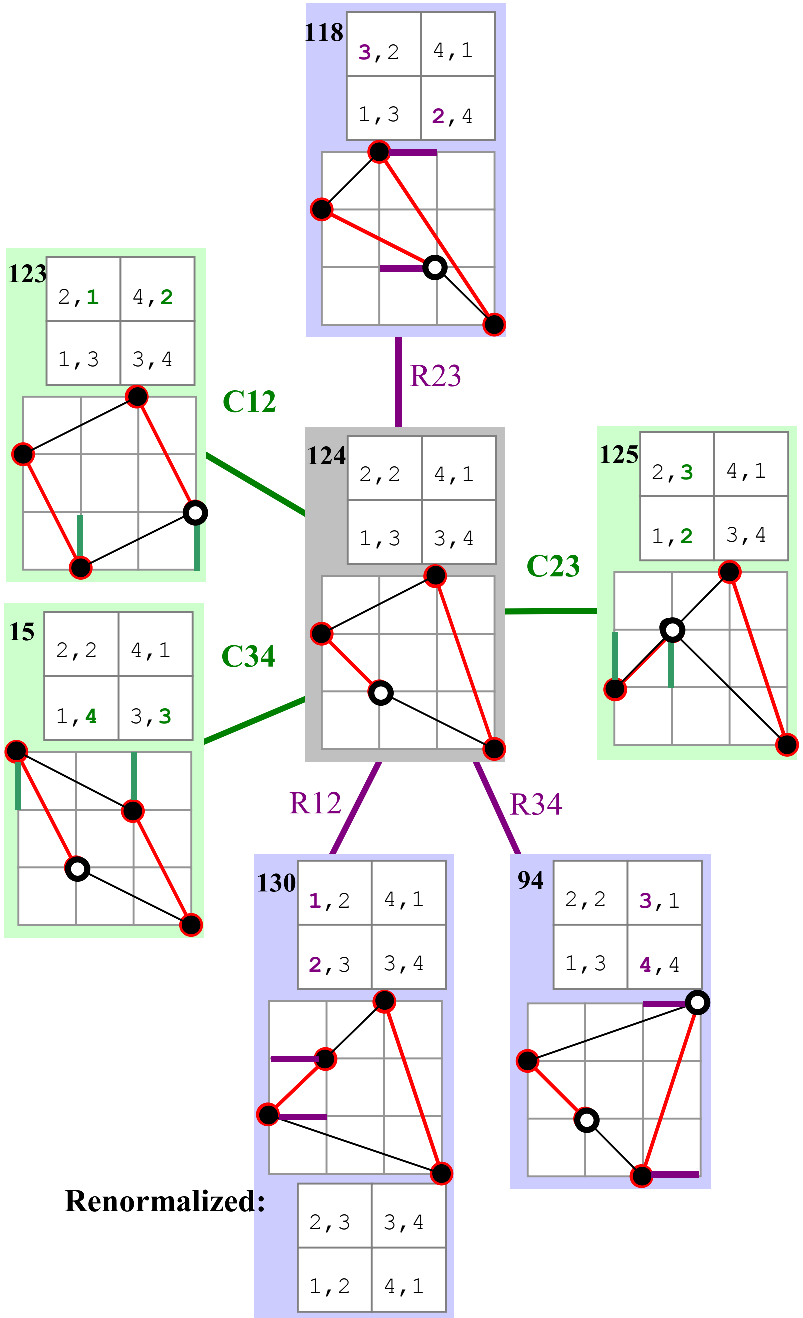



Figure 1

Figure 1: The Neighbourhood of game 124

| Game 124 | C1 | C2 |
|---|---|---|
| R1 | 2, 2 | 4, 1 |
| R2 | 1, 3 | 3, 4 |

### 2.2.1 *Inferior Neighbours*

The neighbourhood N(124) contains the PD, which gives us our first result derived by construction using the topological approach: Sharing the neighbourhood of the PD, there is at least one other game with a strongly Pareto dominated and unique Nash equilibrium. That result demonstrates that exploring the neighbourhood of an interesting game can be productive.

The second result is that $C_{23}$(124) yields another game, 125, with a strongly Pareto-dominated unique Nash equilibrium. We now have a connected region containing games like the PD. We can easily show that the region contains seven games[4], only one of which, the PD, is symmetric. Asymmetric members of the PD family may be more likely to occur than the PD. Only the symmetric member of the family has been studied[5].

### 2.2.2 *Strange Neigbours*

Row swaps from Game 124 yield additional insights. One neigbour, $R_{12}$(124) = 130 has no Nash equilibrium in pure strategies. It will turn out that the games with no Nash equilibria also form a connected set. Furthermore, the family of games with inferior equilibria lies on the boundary of this important group of games.

Another neighbour, $R_{34}$(124 )= 94, has two Nash equilibria. It belongs to another group, as we show below, yielding the surprising fact that a game with no pure strategy equilibrium is just two minimal steps from a game with two equilibria.

---

[4] The set of games that belong to the PD family is {PD, C34(PD), C23(C34(PD)), C12 (C23(C34(PD))), R34(PD), R 23(R 34(PD)), R12 (R23(R34(PD)))} =(15, 57, 63, 69, 124, 125, 126}. Figure 15 situates these games in the topological space of the 2x2 games.

[5] We examine the family of games that the Prisoner's Dilemma belongs to in detail in Chapter 4 of "The Simplest Game", our unpublished monograph on the 2x2 games..

### 2.2.3 *Symmetric Neighbours*

Another intriguing fact emerging from an examination of this small neighbourhood is that Game124 has two symmetric games as neighbours (15, the PD and 94). The figure shows that it is possible to make one symmetric game, the PD, into another symmetric game by the sequence of operations $R_{34}$, $C_{34}$. We call a combined operation such as $S_{34}$ / $R_{34}(C_{34}(g)) = C(_{34}(R_{34}(g))$, in which the same swap is made for the row and the column players, a symmetric operation. A symmetric operation preserves symmetry if it is present initially. It turns out that the symmetric games form a subspace under symmetric operations[6].

This brief look at N(124), the neighbourhood of a single game, shows how the topological approach generates interesting relationships and conjectures.

## 3 Constructing the graph of the 2x2 games

We now develop the complete topological structure of the 2x2 games as the graph of the network generated by the set of swap operators. Beginning with any 2x2 strict ordinal game, repeated applications of the swap operators generates a graph that includes every conceivable strict ordinal 2x2 game exactly once[7].

It is convenient to develop the topological structure in a series of steps. Beginning with a single game and a single swap operator we generate a cycle of two games. We then add an additional operator to generate a larger cycle and continue in this way until we have a structure that includes all and only the known 2x2 games. This progressive concatenation of operations introduces features of the topology in an orderly sequence. It also allows us to present the ordering and numbering conventions we use.

---

[6] The subspace of symmetric games is a rather remarkable object we explore elsewhere.
[7] 144 games are produced by this method, not 78.

## 3.1 *Game 1 and $C_{12}$*

We begin arbitrarily with the following game, which we label Game 1.

| Game 1 | C1 | C2 |
|---|---|---|
| R1 | 4, **2** | 3, 3 |
| R2 | 1, **1** | 2, 4 |

Game 1 is symmetric, which simplifies the presentation. Column's payoffs increase clockwise from the lower right and Rows counter-clockwise. The operation $C_{11}$ , which exchanges the values in bold type ,yields game 2: $C_{12}(1) = 2$.

## 3.2 *Concatenating operations*

Operations may be concatenated. Concatenating a single operation does not produce a third game. $C_{12}(2) = C_{12}(C_{12}(1)) = 1$. The operators are cyclic in this sense. Individual operators are self-inverse. The general observation is that

$$X^{k}ij(g) = g, \quad \text{k even.}$$

Concatenating equivalent orthogonal operations, say $C_{12}$ and $R_{12}$, yields a new game, which we label Game 8.

$$R_{12}(C_{12}(1)) = R_{12}(C_{12}(1)) = 8,$$

Concatenating the combination returns us to game 1:

$$R_{12}(C_{12}(R_{12}(C_{12}(1)))) = R_{12}(C_{12}(8)) = 1.$$

Concatenation of a pair of orthogonal operations in general produces a cycle of four games. The entire network structure can be built up by linking 4-game cycles.

## *3.3 Concatenating overlapping operations*

Concatenated swaps for one player also produce closed subsets of games. $C_{12}$ and $C_{34}$ generate sets of four games. However, $C_{12}$ and $C_{23}$ behave differently because both manipulate the same payoff, 2. Concatenating these operations is not commutative: $C_{12}$ ($C_{23}$ (g)) ≠ $C_{23}$ ($C_{12}$ (g). For example, $C_{12}$ ($C_{23}$ (1)) = 3 but $C_{23}$ ($C_{12}$ (1)) = 6.

Furthermore, repeatedly applying $C_{12}$ and $C_{23}$ produces sets of six games connected in a cycle. For example, from game 1, $C_{12}$ and $C_{23}$ generate the set {1,2,3,4,5,6}. For game 1, column's payoffs increase clockwise from the lower left; $C_{12}$ and $C_{23}$ produce the following cycle of column payoffs:

1234 -> **21**34 -> **3**1**2**4 -> 3**21**4 -> **23**14 -> **1**3**2**4 -> 1**23**4

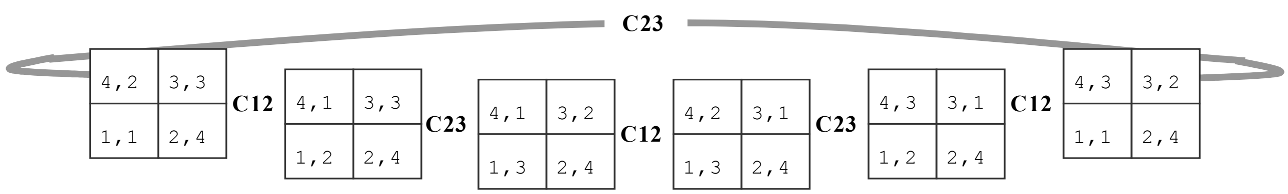



**Figure 2: A 6-game cycle**

Since concatenating swaps using this functional notation quickly becomes unwieldy, it is convenient to introduce abbreviations. Let

$$C_{23}C_{12}(g) = C_{23}(C_{12}(g))$$

and let

$$[C_{23}C_{12}]^2 (g) = (C_{23} C_{12} (C_{23}(C_{12}(g)))).$$

It is easily verified that

$$[C_{23}C_{12}]^3 (1) = C_{23} (C_{12} (C_{23} (C_{12} (C_{23} (C_{12} (1)))))) =1,$$

which is to say, repeating the swaps $C_{12}$ and $C_{23}$ three times results in a closed loop of six games.

We indicate all possible concatenations of $C_{12}$ and $C_{23}$ with the notation for regular expressions: $[C_{12} + C_{23}]^*$. The set $[C_{12} + C_{23}]^*$ (1) is the cycle of games accessible from game 1 with the swaps $C_{12}$ and $C_{23}$.

The games in this cycle share two properties. First, the payoffs for the row player are unchanged since only column swaps have been used. Second, the location of the highest payoff for the column player is constant, since neither $C_{12}$ nor $C_{23}$ affects the highest payoff.

We can construct similar cycles using $C_{23}$ and $C_{34}$ where the payoff of 3 is manipulated by both swaps. The 34 swaps differ from 12 swaps in that they reverse the preference ordering over the two most preferred outcomes. The resulting cycles consist of games in which the location of the **lowest** payoff for the column player is invariant. As a result, $X_{34}$ swaps are most likely to affect the equilibrium in a game. There is an economically interesting sense in which loops for the form $[C_{12} + C_{23}]^*$ are more closely related than loops of the form $[C_{23} + C_{34}]^*$: the payoffs most likely to form a Nash equilibrium are least likely to be affected by form $[C_{12} + C_{23}]^*$. We use this observation in selecting a standard representation of the 2x2 games.

## *3.4 Layer 1: $[X_{12} + X_{23}]^*$*

Clearly every game is one of six games forming a closed loop $[C_{12} + C_{23}]^*$. The same argument using row swaps shows that every game will also be a member of another closed loop of six games. In each loop produced by row swaps, the column player's payoffs are constant and the location of the highest payoff for the row player is constant.

Any loop produced by column swaps will intersect six transverse loops generated by row swaps. But since row swaps change the payoffs for the row player and column swaps change the payoff for the column player, the order that a row and a column swap is applied has no effect: RijCkl = CklRij. It follows that the transverse loops form a surface consisting of 36 games. Every 2x2 game is a member of a surface produced by the concatenation of operations, $[X_{12} + X_{23}]^*$. We will show below that there are 4 such surfaces.

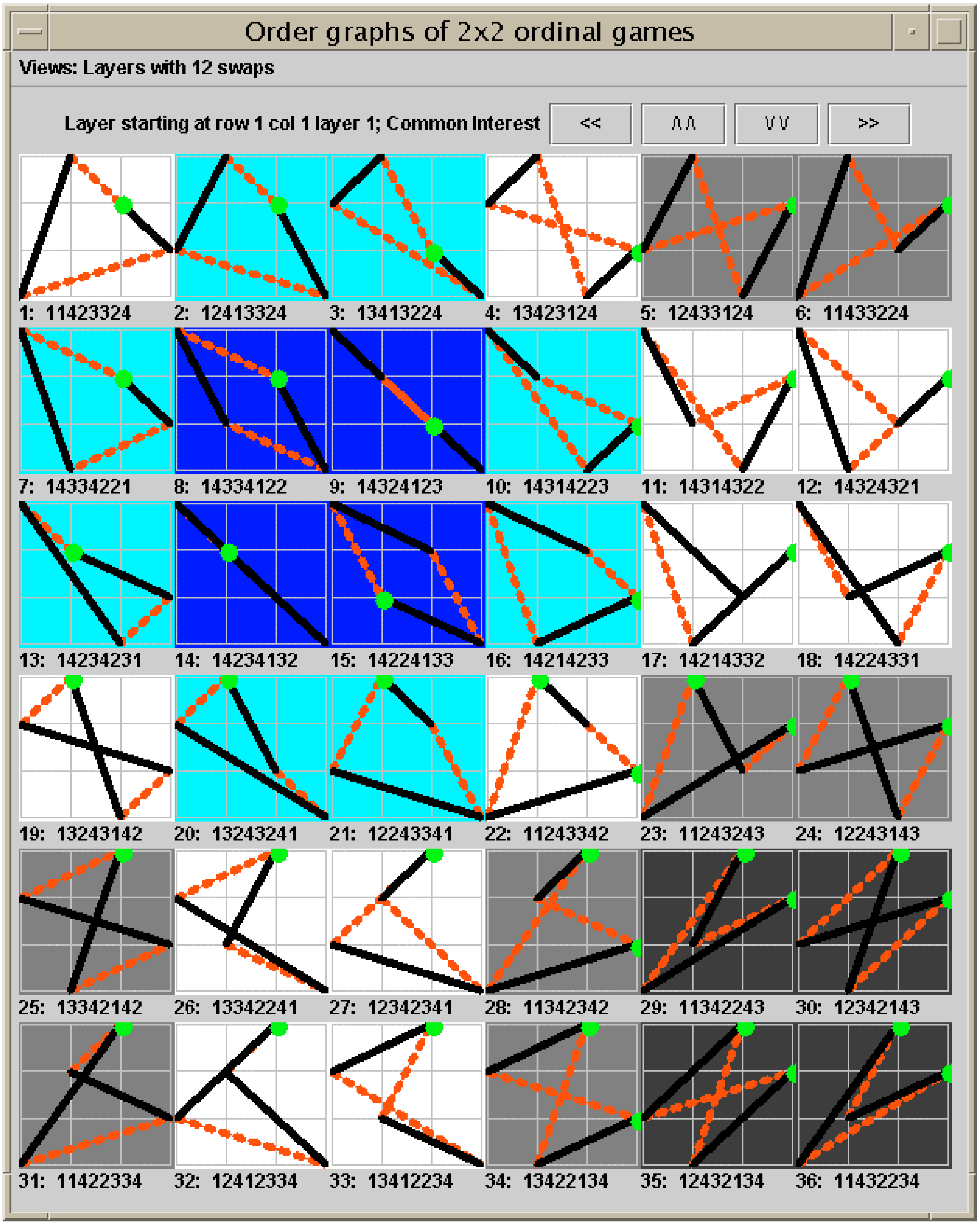



**Figure 3: Layer 1 - a 36-game topological subspace**

The group of 36 games $[X_{12} + X_{23}]^*(g)$, can be represented as a 6x6 grid as in Figure 3[8], which shows the order graphs[9] for each game in the surface games $[X_{12} + X_{23}]^*(1)$. We call the loops produced by swapping payoffs for the column player (column swaps) "rows" since the row player's payoffs are invariant. The loops produced by swapping payoffs for the row player are "columns". Column swaps can be seen as moves along a row to another column.

Since the entire topological space can be reached with any five swap operations, we might say that any five operations span the space. Restriction to four operations or fewer is necessary to produce a subspace. For each game in this subspace, 4 of six possible neighbours are in the subspace. The number of neighbours in the subspace is a measure of the connectedness or density of the subspace. The most dense subspace is 4-connected. Density declines with the number of operations excluded. Surfaces such as those illustrated in Figure 3, with 36 games, are the largest of the densest subspaces.

Adjacent games in Figure 3 are neighbours. Each game has only four neighbours on the surface. Moving from left to right across the surface the operators $C_{12}$ and $C_{23}$ alternate, beginning with $C_{12}$. The perimeter of the diagram consists of $X_{23}$ swaps. The red dotted lines represent Nash sets [10]for the row player.

## *3.5 The topology of Layer 1*

The cyclic property of the swaps implies a specific topology for the surface. Games at the left edge of each layer are neighbours of the games in the same row on the right edge. To show this, in Figure 4 we roll the layer to form a

---

[8] There are two more neighbours for each game that result from swap operations $X_{34}$ that do not appear on these surfaces.

[9] Order graphs are simply the strategic form for ordinal games presented in a discrete payoff space.

[10] We define a Nash set as the points in payoff space accessible given the decision of the other player. It is what Greenberg calls, in The Theory of Social Situations, the inducement correspondence for the Nash situation

cylinder. Since the top and bottom layers must also be joined, the cylinder must be bent so that the ends meet.

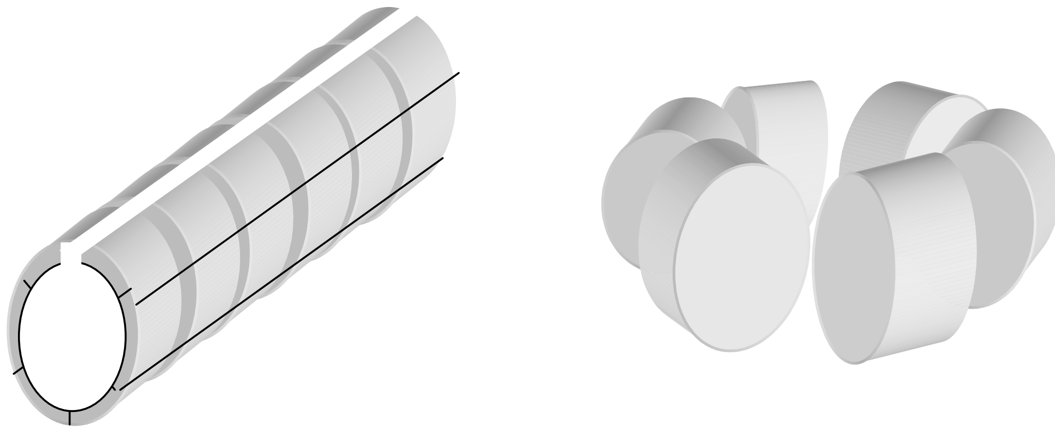


Figure 4: Rolling the plane surface into a torus

As a result, each layer forms a torus[11]. This toroidal structure is one of the major topological features of the 2x2 games[12]. Note that the description is incomplete because we have not introduced links between the layers that are produced by $X_{34}$.

### *3.5.1 Some features of the torus [$X_{12}$ + $X_{23}$]*(1)*

The region represented by this torus has several features that deserve attention. They are easiest to describe using the 6x6 in Figure 3, but they apply to all four surfaces that can be generated with games [$X_{12}$ + $X_{23}$]*(1). We call this configuration with game 1 in the upper left the dominant-strategy configuration[13].

---

[11] In general orthogonal cyclic operations generate a torus.

12 Calculating the Euler number for the network of games generated when each game is considered a vertex and the links to neighbours are considered edges provides another way to test for the nature of the surface .The Euler number is a topological constant for networks. If a network has an Euler number of zero it can be drawn without crossing edges on as torus but not on a sphere. Consider one surface generated Xij*(G) for j<4. The surface contains 36 games, or vertices. It has 4x36/2 = 72 edges, since every edge is shared with one other game. Every game is adjacent to 4 faces and every face has four vertices, soothe number of faces is 36. the Euler number is then E = V + F – E = 36 + 36 – 72 = 0.

13 The dominant strategy configuration is the easiest configuration to relate to other classifications. Alternative configuration can be produced by starting with any game numbered from 2 to 36 in the upper left corner.

In the nine games in the upper left, both players have a dominant strategy. For the nine games in the lower right, neither player has a dominant strategy. For the games in the upper right, only the row player has a dominant strategy. The row player therefore has a dominant strategy for the top three rows of the surface. Similarly, the column player has a dominant strategy for the three columns on the left. The presence of a dominant strategy for even one player in a 2 player strict ordinal game ensures that the games in the first three rows and the first three columns are dominance solvable and can have only one Nash equilibrium.

The regions with zero, one and two players with dominant strategies form the "orders" identified by Rapoport, Guyer and Gordon (1976). The categories identified by Rapoport and Guyer, Rapoport, Guyer and Gordon, and Brams turn out to have a basis in the topology induced by preferences.

The games in the lower right quadrant of this surface all have two Nash equilibria. They are in fact variations on the Battle of the Sexes, games with two Pareto efficient outcomes that differ distributionally[14]. That these games appear as a group is an interesting topological feature of the 2x2 games. A third feature revealed in Figure 3 is that the six games on the diagonal, {1, 8, 15, 22, 29 , 36} are all symmetric. There are only 12 symmetric games, six of which appear on layer 1. Game 15 is the Prisoner's Dilemma, and Game 22 is Chicken. Each symmetric game is two swaps from another symmetric game. The symmetric games are all connected by symmetric operations and in fact form a subspaces under the symmetric operations.

## *3.6 The four-layered torus*

For a given row in $[X_{12} + X_{23}]^{*}(1)$, which we call Layer 1, the payoffs for the row player are invariant. Furthermore, the position of the most preferred outcome for the column player is also invariant. But the most preferred outcome, Rank =4, can appear in any of the four cells of the payoff matrix. It follows that there are

14 Rapoport, Guyer and Gordon cal games 29 and 36 "Leader" and "Hero".

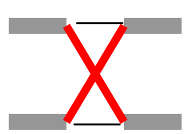
four separate cycles of six games associated with any payoff configuration for the row player. Each of these cycles appears in a separate layer.

| **Swap** | $X_{12}$ | $x_{23}$ | $x_{12}$ | $X_{23}$ | $X_{12}$ | $X_{23}$ |
|---|---|---|---|---|---|---|
| **Layer 1:** | 1234 -> 2134 | -> 3124 | -> 3214 | -> 2314 | -> 1324 | -> 1234 |
| **Layer 2:** | 1243 -> 2143 | -> 3142 | -> 3241 | -> 2341 | -> 1342 | -> 1243 |
| **Layer 3:** | 1423 -> 2413 | -> 3412 | -> 3421 | -> 2431 | -> 1432 | -> 1423 |
| **Layer 4:** | 4123 -> 4213 | -> 4312 | -> 4321 | -> 4231 | -> 4132 | -> 4123 |

There are, therefore, four layers of 36 games, for a total of 144 games.

### *3.6.1 Links between layers*

We call the games that are aligned vertically a "stack". To see the relationship between the layers, notice first that to get from the first stack on the left to the second always requires an $X_{12}$ operation to get from the second to the third requires an $X_{23}$ and from the third to the fourth requires another $X_{12}$. The order of operations is the same in each layer. We can move from the first game in layer 1 to the second game in Layer 2 by performing an $X_{34}$ operation on 1234. The $X_{34}$ swap moves us between layers. The $X_{34}$ swap applied to the first game in the second layer leads to the second game in the first layer. The first two games in the first two layers are linked this way:



$X_{34}$ does not always lead from the first to the second layer. In fact, $X_{34}$ produces the pattern of links between layers shown in Figure 5. Layer 1 has 6 “seams" joining it to other layers. One horizontal and one vertical seam link layer 1 to Layer 2. Another horizontal and vertical pair links Layer 1 to layer 4, and the last pair links Layer 1 to Layer 3. Figure 5 identifies all the ties between the four layers.

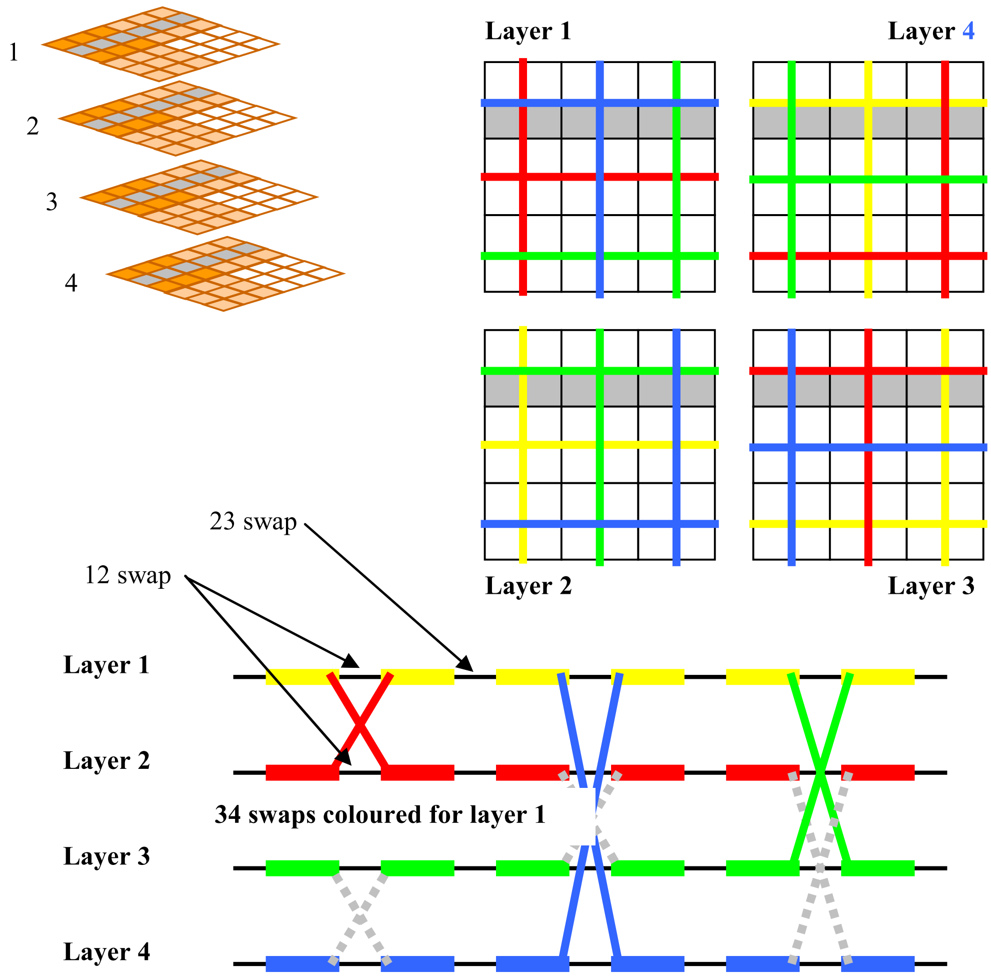



Figure 5: Location of 34 swaps and layer reached

The four interconnected layers of 36 games form a 4-layered torus. Each layer can be generated separately using $C_{12}$, $R_{12}$ and $C_{23}$, $R_{23}$ swaps, and the four layers are linked by the $C_{34}$ and $R_{34}$ swaps. The links between the layers produce topological features of some interest. They are best approached indirectly.

### *3.6.2 The structure of a stack*

We have seen above how $C_{12}$ keeps us within a layer and $C_{34}$ moves us between layers. Combining the two moves us to another layer in the same stack. The two operations combined are equivalent to a row operation, a column operation, or both on the payoff matrix for the column player. Three examples are shown in Figure 6

Equivalent to a Column operation

| | | | | |
|---|---|---|---|---|
| 1 | 2 | $C_{12}BC_{34}$ | 2 | 1 |
| 4 | 3 | → | 3 | 4 |

Equivalent to both

| | | | | |
|---|---|---|---|---|
| 1 | 3 | $C_{12}BC_{34}$ | 2 | 4 |
| 4 | 2 | → | 3 | 1 |

Equivalent to a Row operation

| | | | | |
|---|---|---|---|---|
| 1 | 4 | $C_{12}BC_{34}$ | 2 | 3 |
| 2 | 3 | → | 1 | 4 |

**Figure 6: The structure of a stack**

The games in a given stack exhibit a peculiar form of invariance. Within a stack, payoffs that are diagonally opposite remain diagonally opposite. There are exactly four such configurations for any given payoff pattern, corresponding to the 4 layers There are exactly six possible initial positions, corresponding to the six columns.

What can such an odd pattern of invariance correspond to? The change is equivalent to re-labelling the strategies from the point of view of the column player. In the absence of information about the payoffs of the other player, these four configurations are equivalent. The two-swap sequence, $C_{12}C_{34}$, produces a game which is the same as the original game for the column player **if the column player has no information about the other player's payoffs**. This association with the original game leads us to identify $C_{12}C_{34}(1)$ as game 37, the first game on the layer immediately below layer 1.

## 3.7 Pipes:

Figure 5 shows the linkages between the layers. These produce another set of toroidal structures analogous to the layers that we have already described. If we begin with any game, $X_{12}$ produces a cycle four games that differ only in the order of the least preferred elements. These cycles we call “tiles” because the structure of the games can be built up out of them. There are 9 such tiles on each layer. Figure 7 illustrates the tiling of layer 1. It is convenient to label the tiles in each layer. For reasons that will become clear, we call a stack of tiles a pipe.

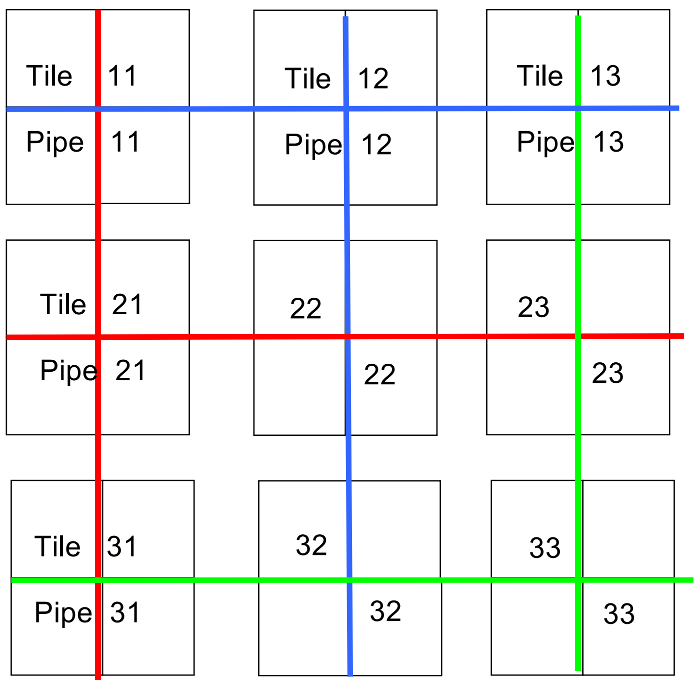



**Figure 7: Tiling layer 1 with 4-game tiles**

Notice that the games in each tile are linked to games on another layer by $X_{34}$ swaps. The color of the lines indicates the layer that $X_{34}$ connects layer 1 to (red to layer 2, green to 3, blue to 4). There are two cases. Either $R_{34}$ and $C_{34}$ connect to the same layer or they do not.

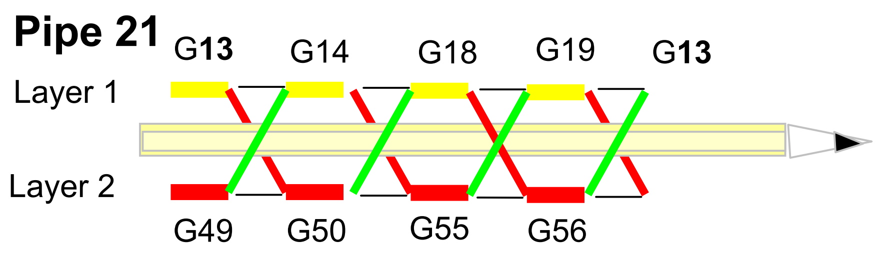



**Figure 8: Eight-game pipe 21 (12)**

When the $X_{34}$ swaps connect to the same layer they produce a cycle of 8 games. This occurs, for example with games in pipe 12, pipe 21 and pipe 33. These six objects are the smallest dense subspaces.

To understand the structure $[X_{12}+X_{34}]^*(13)$, imagine running a rubber pencil under the green links between layers and over the red links. The four games in layer one form a loop, as do the four games in layer 2, so bend the pencil around and join the ends so that it forms a torus. The links between the games in $[X_{12}+X_{34}]^*(13)$ form a network with no crossing edges on the torus. In other words, the subspace, $[X_{12}+X_{34}]^*(13)$, is itself a torus joining layers 1 and two.

Since links between games on layer 1 and 2 are topologically equivalent to links between games on layer 1, layers 1 and 2 must be part of the same surface. It follows that the torus which we have just identified surrounds a sphincter-like connection between layers 1 and 2. The addition of one feature like this, connecting two torii, produces a two-holed torus. Since there are six such objects, the four-layered torus described above is seen to be at least a seven-holed torus when the six 8-game pipes are taken into account. Figure 9 illustrates the structure schematically, ignoring the important fact that the pipes are distributed symmetrically on the torii for each layer[15].

[15] We are still puzzled by a feature that Figure 9 does not capture: each of the pipes is "thick" in the sense that the wall of the pipe is itself a torus.

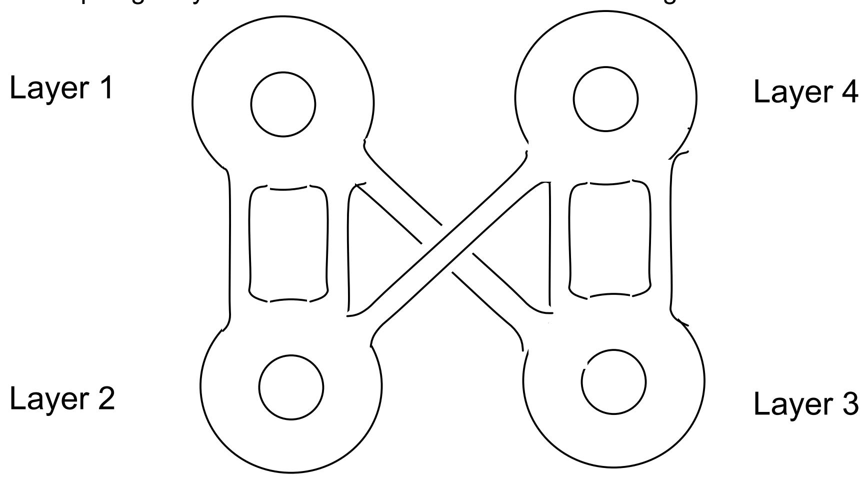



**Figure 9: Partial view - a seven-holed torus**

Now consider the remaining four distinct regions produced by $X_{12}$*$X_{34}$*. These appear as pipes associated with tile 11, 13(31), 22, and 23(32). When $R_{34}$ and $C_{34}$ swaps connect to different layers all sixteen games in the four stacks involved are connected.

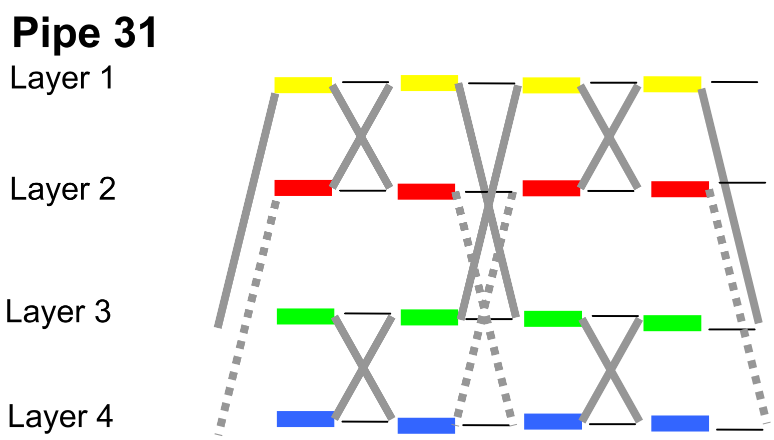



**Figure 10: The 16 game pipe 31 (13)**

It can be shown that these also form torii, but that these torii link all four of the [$X_{12}$+$X_{34}$]*(*g*), layers. An argument similar to the one that shows that each of the 8-game torii adds a hole to the concentric torii above formed by the 4 layers shows that each of the 16-game torii linking all four layers adds four holes. There are 6 16-game torii. The entire structure is thus at least a 31-holed torus[16].

# 4 Topologically based classifications

To this point we have concentrated on developing a description of the topological space of the 2x2 games. In this section we will demonstrate that the subspaces provide an interesting framework for classifying the games.

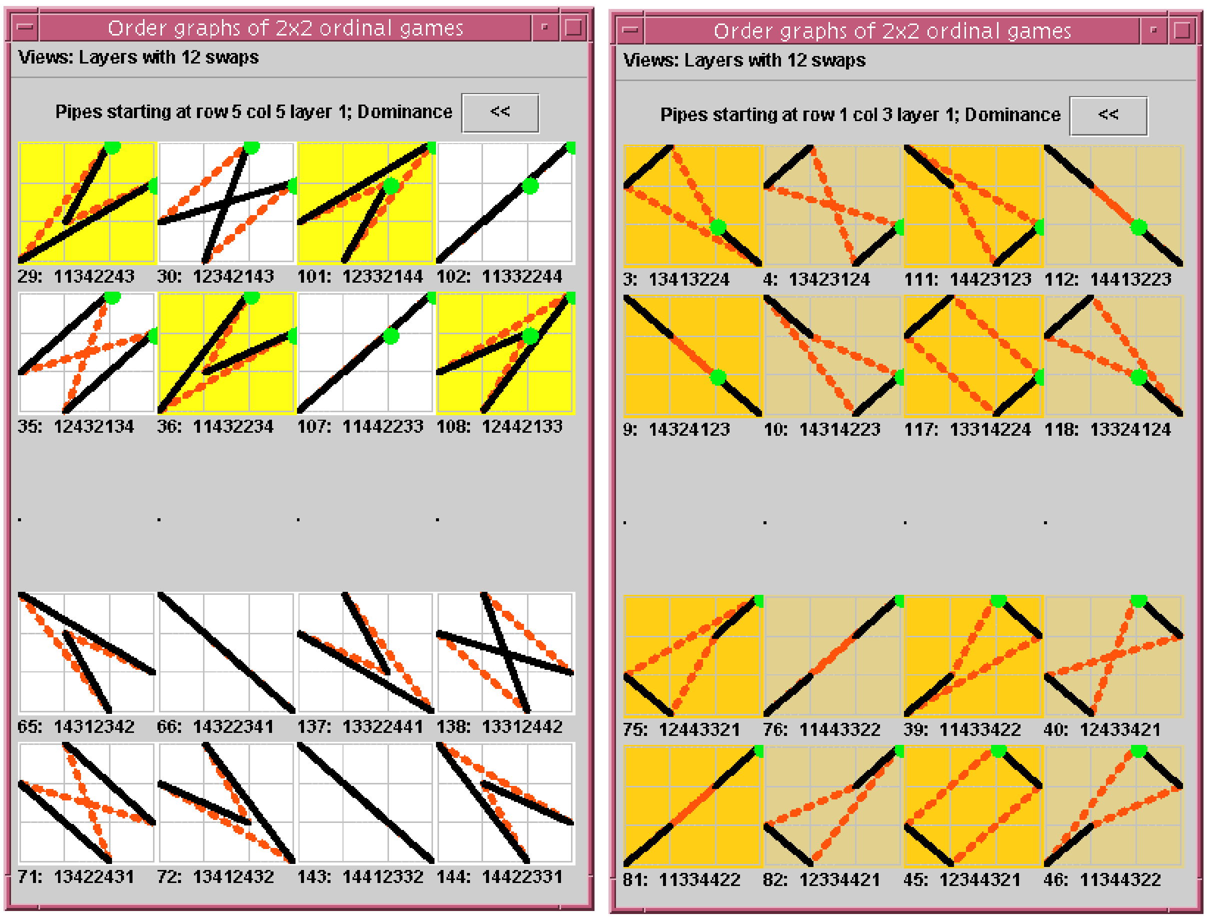



**Figure 11: Four eight-game torii, $X_{12}*X_{34}*$**

---

[16] 7 + (4x6) = 31. This is a remarkable and, as nearly as we can tell, a profoundly useless discovery! The uncertainty at this point arises because we have not completely dealt with the 8-game pipes. One of us (Goforth) conjectures that each adds 2 more holes, making the grand total 43, but we do not have a proof yet.

## *4.1 What do we have in our pipes?*

### *4.1.1 The 8-game pipes*

There are four distinct eight-game torii generated by $[X_{12}+X_{34}]^*(g)$. The two sets on the right of Figure 11 are from tile 21. They are reflections of those at tile 12. On the left are the order graphs for the two torii at tile 33. Inspection of the order graphs reveals that the games in each stack are closely related and that there are relationships between the stacks. The games at 33 either have two or no Nash equilibria.

The top set on the left, which includes games from layers 1 and 3, are either variants of the battle of the sexes, with two efficient but distributrionally distinct equilibria, or pure coordination games. In all of these games, every Nash set is positively sloped. These are games of pure common interest. Any change that benefits one player benefits the other. It is remarkable, in our view, that such a concentration of interesting games should be found in one of the smallest dense subspaces of the 2x2 games.

The games in the bottom set on the left are from layers 2 and 4. They have no Nash equilibria. Again, it is remarkable that so many of the games with no Nash equilibria should be found in a single small subspace and that only games of this nature are found in this subspace. Unlike the previous set, these are games of pure conflict: a move that benefits one player makes the other worse off. Graphically, every Nash set is negatively sloped. Notice that the games in this group can be produced by rotating the order graphs for the games in the previous set.

Each game in the torus joining layers 2 and 4 at 33 has a reflection in the region. Surprisingly, there are only two pairs of reflections in the torus joining layers 1 and 3.

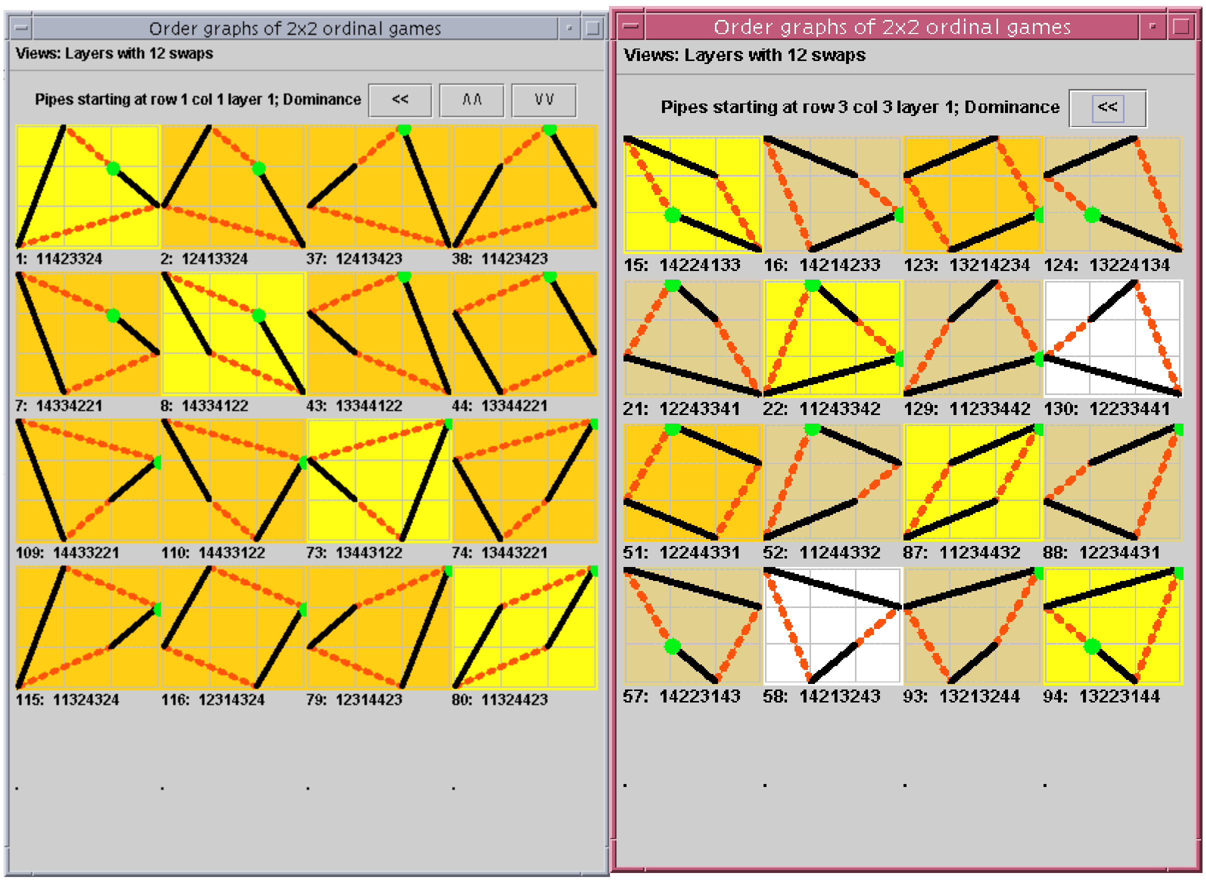
The games on tile 21(12) have the property that they have the same payoffs as those in on 33 but they are “wired” differently. As with the games on

33, the games in one set can be produced by rotating the order graphs for the games in the other set. Unlike the games on 33, all the games on 12(21) have a unique Nash equilibrium.

### *4.1.2 The 16-game pipes*

Figure 12 shows the 11 and 22 pipes. Each can be rolled into a torus as illustrated in Figure 4. [17]

Figure 12: The 16-game pipes, 11 and 22, the "microcosm", which includes the PD.




[17] The eight-game blocks illustrated in Figure 11 do not roll as neatly. Each consists of two "tiles". To produce a torus the simplest way it is necessary to repeat the tile on the right below the tile on the left, and the tile on the left below the tile on the right. The result is a 16-game, 4-tile surface, reflect the fact that two circuits must made around each pipe to traverse every edge.

Each game in pipe 11 can be described as the result of flipping a game in pipe 22 around the positive diagonal of the payoff space. The negative diagonal in both sets of games consists of symmetric games. Games in the upper right are reflections of games in the lower left in each set.. If the players are identical then the reflections are equivalent and there are only 10 distinct games in these pipes. Notice, however, that reflections in this sense are not neighbours – they are not topologically near each other. Games in the upper right, for example, can be transformed into their reflections with a minimum of 2 swaps and may require 4.

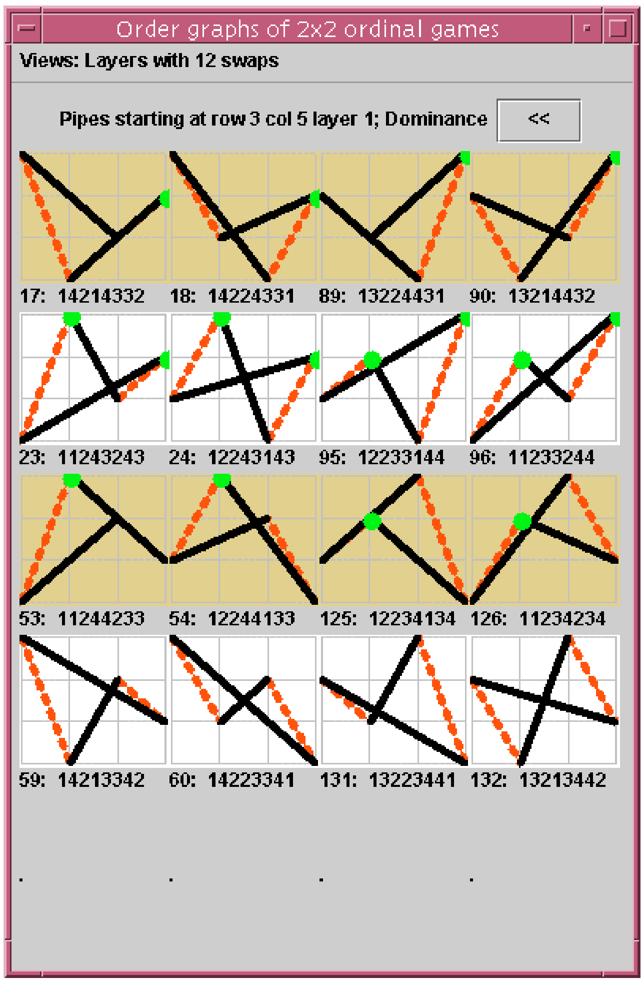



Figure 13: The 16-game pipe P23

The subspace represented in the right panel of Figure 12 is a microcosm of the entire space of 2x2 games. It includes the Prisoner's dilemma and two other games with inferior equilibria, two games with two equilibria (one is

“Chicken”), two games with no equilibria, four “no conflict” games, and seven games with rank-inefficient Nash equilibria.

Figure 13 shows the games in the pipe 23 (reflected by 32). This region includes four games with no Nash equilibrium and four games with two Nash equilibria. The latter can be seen as asymmetric Battle of the Sexes and coordination games.

### *4.1.3 The “no conflict" region*

Layer 3 is distinguished by a feature which seemed especially important to Rapoport and Guyer (1966), Rapoport, Guyer and Gordon, (1976), and Brams (1992?). Notice that in every stack there must be a game with the payoff combination (4, 4). Furthermore, since the operations that produce a layer do not include the 34 swaps, all such games must be in the same layer. In our numbering system, layer 3 includes all 36 games with the payoff combination (4, 4). Rapoport and Guyer (1966), Rapoport, Guyer and Gordon, (1976), and Brams (1992?) describe this set of games as the “no conflict games”. In the RGG typology these games constitute a “phylum”, which is the highest level in their classification. This is case where the typology has a topological basis.

The other Rapoport, Guyer and Gordon phyla consist of “pure conflict games” and ”mixed motive games”.

By combining the partition by dominance and the partition Rapoport and Guyer/ Rapoport, Guyer and Gordon actually produced a cross-classification of the games by a variety of features, none of which has logical priority.

## 5 Typology and topology

We have derived the topology of the 2x2 games in some detail. Our argument is that the topological approach provides the natural basis for the study of and classification of the 2x2 games. It essentially replaces previous typologies.

The major observation is that related games are associated with distinct regions in the topological space. A striking example is provided by the games identified by Rapoport and Guyer as “no conflict” games. This diverse collection of games which includes several co-ordination games and 9 games with dominant strategy equilibria was identified by the presence of a single feature of the payoff structure, (the presence of a (4, 4) payoff). The entire group was shown to be the subspace generated by $[X_{12}+X_{34}]^*(g)$, operating on any game with the payoff combination (4, 4).

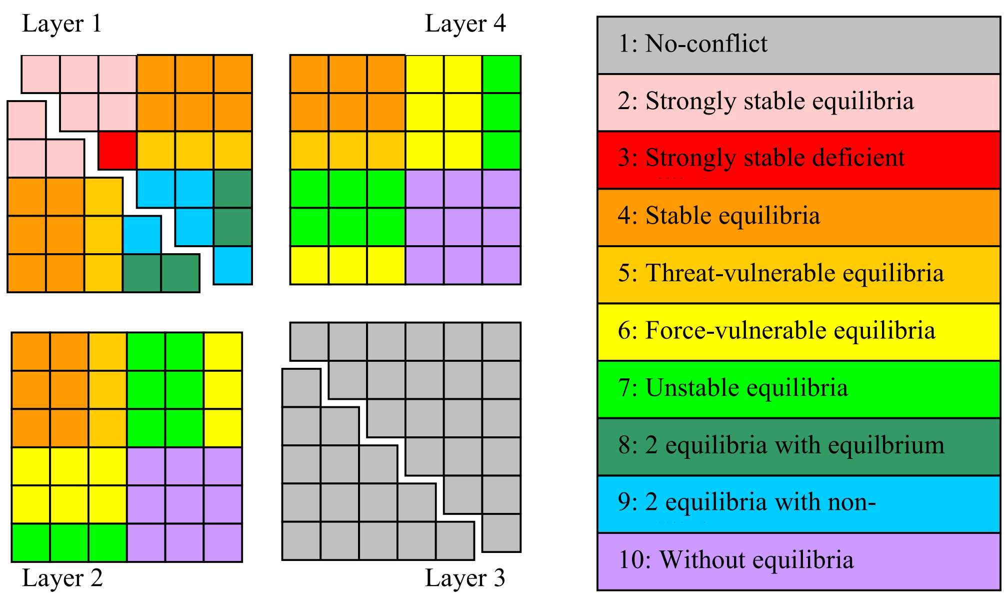



Figure 14: The Rapoport and Guyer typology

Rapoport and Guyer cross-classified games according to whether two, one, or no players have a dominant strategy[18]. We have shown that games with dominant strategy equilibria and games in which no player has a dominant

---

[18] Rapoport and Guyer did not apply this cross-classification to the no-conflict games, but Rapoport, Guyer and Gordon repaired this inconsistency in the earlier work. There is some conceptual difficulty in the way that Rapoport and Guyer identify a cross-classification, or matrix structure, with a typology, normally associated with a tree structure. The .

strategy[19] form connected regions. To a significant extent the topological analysis validates the work of previous analysts.

Not all the distinctions made in earlier works have a foundation in the topology, however. Figure 14 displays the Rapoport and Guyer classification in our standard representation.

Rapoport and Guyer chose to eliminate reflections, on the assumption that players are identical. The assumption has no topological justification. The games they retained do not represent a connected region and their game numbering system is not systematic over the topological space. Figure 13 includes the 78 games they counted and their reflections. Eliminating layer 2 and games below the diagonal in layers 1 and 3 leaves a connected but non-cyclic region containing 78 games equivalent to the Rapoport and Guyer classification. The 78-game structure doesn't support the complete topology.

Notice first that types 2, 3, 4 and 5, plus 6 of the 36 no-conflict games on layer 3 make up the block of games with dominant strategy equilibria

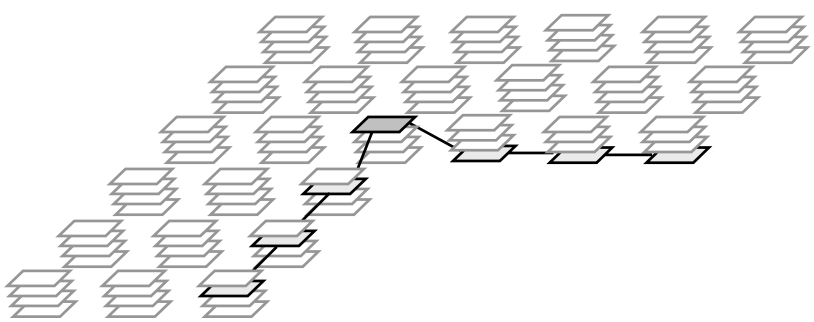


**Figure 15: The Prisoner's dilemma family**

The single “strongly stable, deficient” game identified by Rapoport and Guyer is the prisoner’s dilemma. There are in fact six other games with deficient equilibria that are not identified. . Clearly these topologically and economically meaningful group are not  identified by the Rapoport and Guyer typology. The

[19] Corresponding to Rapoport, Guyer and Gordon’s $D_2$ and $D_0$ respectively.

games are on layer 2 and 4. They form a connected set with the PD as shown in Figure 15.

The remarkable diversity of the 22 pipe, which includes the PD is revealed, but the relationships between the games on layers 1 and 3 through the 33 pipe is completely obscured. Two eight-game pipes separate the games into two groups - those with no equilibrium and those with two equilibria.

The set of symmetric games on the diagonal of layers 1 and 3 are not distinguished by Rapoport and Guyer, although they form a meridian of the topological space. The other meridian of the topological space runs through the eight game pipes and contains the pure common interest games and the game of pure conflict.

# 6 Conclusions

The topology of the 2x2 games is induced by the structure of preferences. The resulting topological space is toroidal, but can easily be disassembled into subspaces that can be represented as surfaces. We have shown that the entire space is torus with at least 31-holes. The holes represent regions of relatively dense connections. The games in these regions show important similarities. There are two types of holes – those connecting two torii and those connecting 4.

The topological approach provides a systematic and economically meaningful basis for classifying the 2x2 games. It is capable of being generalized to larger games[20].

[20] Experience also tells us that the approach is teachable and suitable for undergraduate instruction.

## References

Brams, Steven j.. A Theory of Moves. Cambridge, Cambridge University Press, 1994.

Rapoport, Anatol, Melvin J. Guyer. “A Taxonomy of 2x2 Games,” General Systems. 11 (1966): 203-214.

Rapoport, Anatol, Melvin J. Guyer and David G. Gordon. The 2x2 Game. Ann Arbor. The University of Michigan Press.1976

Greenberg, Joseph. The Theory of Social Situations: an Alternative Game-Theoretic Approach. Cambridge, Cambridge University Press, 1990.

Weise and Noack, “Aspects of topology”, Fundamentals of Mathematics: vol 2, Geometry Ch 16, 1986. English translation of Grundzhge der Mathematcik, second German edition 1967 and 1971.