Conflict, No Conflict, Common Interests, and Mixed Interests in 2×2 Games*

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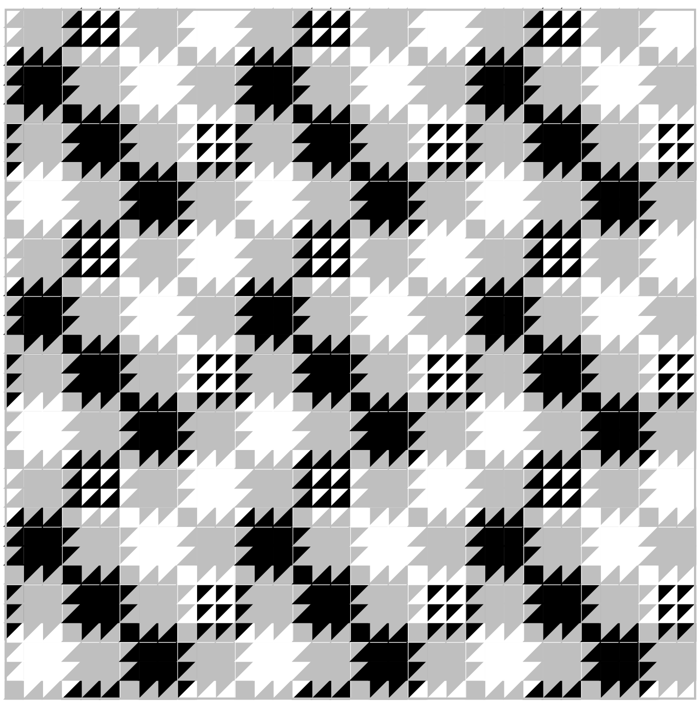

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Abstract

Abstract game theory originated in the analysis of games of pure conflict. By 1958, however, Schelling set out to “enlarge the scope of game theory, taking the zero-sum game to be the limiting case rather than a point of departure”. The other limiting case, in his view, was the coordination game. Classification according to the degree of conflict quickly became standard procedure. In 1966 Anatol Rapoport and Melvin Guyer published a taxonomy of the 2×2 games in which the largest category was No-Conflict Games. In 1976 Rapoport, Guyer and David Gordon revised the classification, making No-Conflict Games, Mixed-Motive Games and Games of Complete Opposition the primary categories. No satisfactory classification of the so-called Mixed-Motive Games has emerged.

We present a systematic classification based on the topology of the 2×2 games. Topological relationships and symmetry both prove useful in identifying games of pure conflict and games of pure common interest. Our topological treatment reveals a class of games, the Type Games, that has not previously been recognized as distinct. In Type Games the players live in different moral universes. One type is never led into temptation; the other is never free of temptation. One needs no moral instruction; the other must be restrained by law. We also provide an improved terminology for discussing conflict in the 2×2 games, a new subdivision of the games, and a pretty map of the relationships among games based on the degree of conflict.

I. INTRODUCTION

Game theory is often marketed as the study of conflict. The titles of books dealing with or using game theory often emphasize conflict: examples include *Game theory: Analysis of Conflict*, by Myerson [8], *Game theory: Mathematical Models of Conflict*, by Jones [5], *Game theory as a Theory of Conflict Resolution*, by Rapoport [11], *The Strategy of Conflict*, by Schelling [18], *Conflict among Nations: Bargaining, Decision Making and System Structure in International Crises*, by Snyder and Diesing [20], and *The Structure of Conflict*, edited by Paul Swingle [21].

Many authors identify game theory with conflict:

> *Game theory is a branch of mathematical analysis developed to study decision making in conflict situations.*
>
> Principia Cybernetica Web [4]

> *Game theory is the interdisciplinary study of conflict.*
>
> Dr. Daniel King [6]

> *Game theory studies formal models of conflict and cooperation.*
>
> Dr. Bernhardt von Strengel [23]

Indeed, game theory originates in the analysis of games of pure conflict: The *Theory of Games and Economic Behavior* is based on a solution to the zero sum games[1]. By 1958, however, Schelling was calling for more attention to non-constant sum games. "Pure conflict, in which the interests of two antagonists are completely opposed, is a special case." ([18] p. 4)

> On the strategy of pure conflict – the zero-sum games – *game theory* has yielded important insight and advice. But on the strategy of action where conflict is mixed with mutual dependence – the nonzero-sum games involved in wars and threats of war, strikes, negotiations, criminal deterrence, class war, race war, price war, blackmail, maneuvering in a bureaucracy or in a traffic jam, and the coercion of one's own children – traditional game theory has not yielded comparable insight or advice.

Schelling set out to "enlarge the scope of game theory, taking the zero-sum game to be the limiting case rather than a point of departure". The other limiting case, in his view, was the coordination game.

[1] Von Neumann and Morgenstern write "While these games are not typical for major economic processes, they contain some universally important traits of all games and the results derived from them are the basis of the general theory of games." ([22] p. 34)

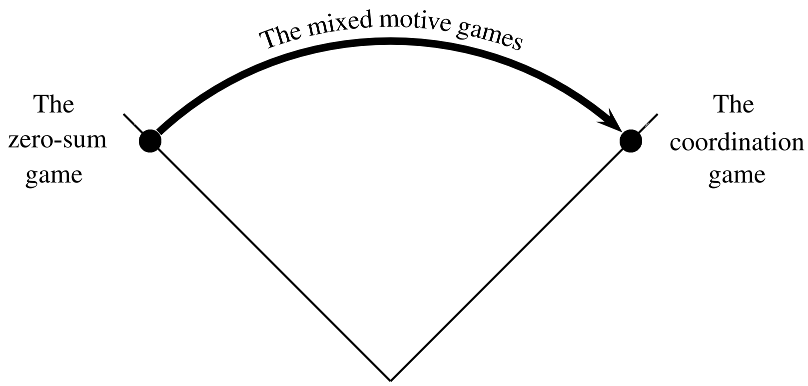


Figure 1. A graphical view of Schelling's classification scheme

> The essentials of a classification scheme for a two-person game could be represented on a two-dimensional diagram... All possible outcomes of a pure conflict game would be represented by a some or all of the points on a negatively inclined line, those of a pure common interest game by some or all of the points on a positively inclined line. In the mixed game, or bargaining situation, at least one pair of points would denote a negative slope and at least one pair a positive slope.([18] p. 88)

A version of the model Schelling describes in the quotation is shown as Figure 1. A more complete model, incorporating some of the results in this paper, appears later as Figure 11.

Schelling himself provided one of the most satisfying discussions of the mixed motive games. By 1970 Anatol Rapoport would write, "It seems to me that the real value of game theory... lies in the subsequent development of the theory beyond the context of the two-person constant sum game."([11]p. 38)

Classification according to the degree of conflict quickly became standard procedure. In 1966 Rapoport and Melvin Guyer published a taxonomy of the 2×2 games in which the largest category was *No-Conflict Games.* In 1976 Rapoport, Guyer and David Gordon [13] revised the classification, making *No-Conflict Games*, *Mixed-Motive Games* and *Games of Complete Opposition* the primary categories.

Even so, relatively few mixed motive games have been examined. In *The Strategy of Conflict*, Schelling explicitly analyzed only seven. Although he suggested a classification scheme, he did not provide a systematic classification and no satisfactory classification of the so-called mixed motive games has emerged.

In this paper we present a systematic classification based on the topology of the 2×2 games[2]. We begin with a simple way to represent the 2×2 games.

[2] The treatment is based on our forthcoming *A New Periodic Table: The Topology of the* 2×2 *games*

		Column Player's Strategies	
		L	*R*
RowPlayer's Strategies	*U*	1,4	3,3
	D	2,2	4,1

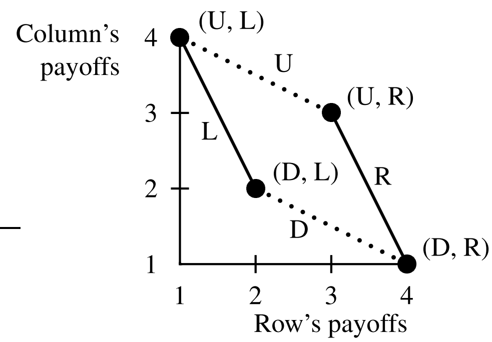


Figure 2. Payoff combinations for the Prisoner's Dilemma

II. Representing the 2×2 games in payoff space

It is convenient to shift analysis from the space of strategies to a space of payoffs[3] where Row's payoffs are plotted on the horizontal axis and Column's on the vertical axis.

In a payoff matrix, each cell represents one possible outcome. Each of the associated payoff vectors appears as a point in the payoff space of the game. Figure 2 shows the Prisoner's Dilemma in strategy space and in payoff space.

We call the set of payoff vectors when one player's choices are fixed an *inducement correspondence for the Nash situation*. The inducement correspondence is a concept developed by Greenberg [2] that is particularly suited to the payoff-space representation. An inducement correspondence is a general term for a set of positions that one player can bring about, or 'induce'. When we refer to "Row's inducement correspondence" we mean the set of outcomes induced *by* the column player *for* the row player to choose from. Row's inducement correspondences for the Nash situation are always columns of the payoff matrix, and Column's are always rows.

The inducement correspondences for Prisoner's Dilemma are shown in Figure 2 by linking the outcomes in each inducement correspondence with a line. Solid lines identify Row's two inducement correspondences (linking alternatives available to Row once Column has chosen) and dotted lines identify the inducement correspondences for the column player[4].

Even without the labels, Figure 2 is a complete representation of a 2×2 game in strategic form. Any 2×2 payoff matrix can be represented as a graph of this sort, and any graph of this sort can be translated into a well-formed payoff matrix.

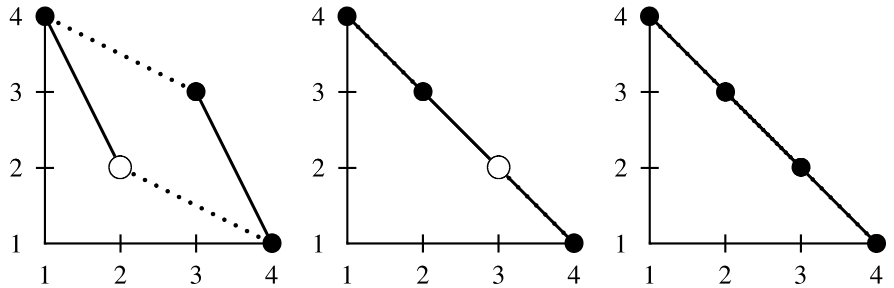
[3] This dual to the familiar matrix representation can provide a more direct and intuitive approach to game theory for students.

[4] Each of the four sides of the quadrilateral in Figure 2 corresponds to a row or a column of the payoff matrix. Strategy names can therefore be used to label the inducement correspondences in 2×2 games, as we have done in Figure 2. When hand-sketching these graphs, a dotted line is harder to draw than a solid line, so we use a double line for Row's inducement correspondences and a single line for Column's.

1,4	3,3
2,2	4,1

1,4	2,3
3,2	4,1

1,4	4,1
3,2	2,3



(a) Prisoner's Dilemma (b) Zero-Sum Nash equilibrium (c) Zero-Sum No Nash equilibrium

Figure 3. Pure conflict games

III. INDUCEMENT CORRESPONDENCES AND CONFLICT

Games of pure conflict, pure common interest and mixed motives can be described in terms of the slopes of the inducement correspondences. In a game of pure conflict every inducement correspondence is negatively sloped. Any action that improves the outcome for one player must make the outcome worse for the other. In a game of pure common interest, every inducement correspondence is positively sloped.

Mixed-motive games, to use Schelling's term, can have one, two or three positively sloped inducement correspondences. There are therefore five levels of conflict for 2×2 strict ordinal games. The situation with two positively sloped inducement correspondences can be further divided into the case in which both players have mixed slopes and the case when one player has positive slopes and one has negative slopes. This latter situation is significantly different from the other mixed motives games, as we will show.

Using inducement correspondences seems natural, but it results in a classification that differs from Schelling's and others'. If a hard distinction is made between zero-sum games and nonzero-sum games, as Schelling does, the Prisoner's Dilemma and the six zero-sum games are distinguished[5]. If the distinction is made instead between games in which all inducement correspondences are negatively sloped and those with one positive slope, as we suggest, the constant-rank-sum games are grouped with the Prisoner's Dilemma and seven other games with negatively sloped inducement correspondences. Arguably the Prisoner's Dilemma repeated in Figure 3*a* is more like the constant-rank-sum game of Figure 3*b*, which has a single Nash equilibrium[6], than this latter game is like the constant-rank-sum game of Figure 3*c* which has *no* Nash equilibrium. The point is that grouping games on the basis of the slope of the inducement correspondences pro-

[5]Note the constant rank-sum games are the ordinal equivalent of the zero-sum games.

[6]The equilibrium outcome, if it exists, appears on the diagram as an open dot.

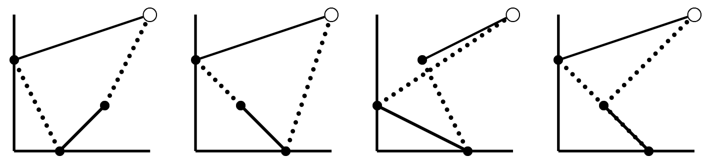


Figure 4. Some Rapoport and Guyer "No conflict" games that are "mixed motive" games based on the inducement correspondence classification

vides a coherent but distinct approach.

The no-conflict games are also treated differently using the inducement correspondence approach. Schelling asked "The zero-sum game is the limiting case of pure conflict; what is the other extreme?" and answered, "It must be the 'pure collaboration' games in which the players win or lose together [...] they must rank all outcomes identically" ([18] p. 84). There are six ordinal games that fit this description. Four have a single Pareto-optimal equilibrium and two have two Nash equilibria, suggesting that the category is not internally consistent.

There are eight other games with four positively sloped inducement correspondences, some with one and some with two equilibria. Are the games with two equilibria and positively sloped inducement correspondences most like other games with two equilibria or most like games in which the players have common rankings? Topologically they are closer to the related Coordination games than to other common ranking games, suggesting that Schelling's classification is externally inconsistent as well.

There is another terminological issue that can muddy the water. Rapoport and Guyer [12], Rapoport, Guyer and Gordon [13], and Brams [1] all describe games in which both players can achieve their best outcome simultaneously as "no-conflict" games. The payoff matrix must therefore contain a cell with the payoff vector $(4,4)$. By this definition, there are 36 "no-conflict" games including ten games with two negatively-sloped inducement correspondences and sixteen with one negatively-sloped inducement correspondence. In other words, most of the no-conflict games are mixed-motive games under Schelling's widely accepted classification. Figure 4 provides four examples.

IV. A MAP OF THE 144 2×2 GAMES

Robinson and Goforth [15] have shown that there are 144 ordinal 2×2 games and that they constitute a topological space under a simple set of rules for transforming games into "similar" games: each game has six neighbours so there are 432 edges joining the 144 games into a connected graph[7]. A convenient representation of this graph is a 12×12 square (Figure 5) where adjacent games differ by only the smallest possible change. Row's payoffs are invariant in the games in each row; Column's payoffs are the same in every game in a given column.

[7]For a complete treatment see Robinson and Goforth [16]

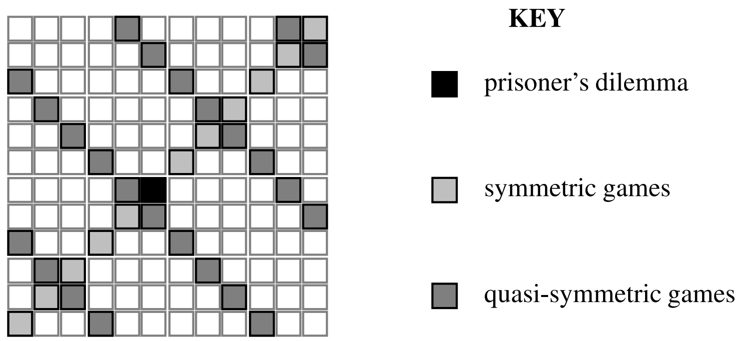


Figure 5. Schematic view of Periodic Tables showing location of PD, Symmetric, and Quasi-symmetric games

The 12×12 layout can be embedded in a simple torus so that games on the bottom row are adjacent to games on the top row and games at the left connect to games at the right. We call the 12×12 layout the Periodic Table of the 2×2 games. 288 of the 432 edges of the graph of the 2×2 games are shown, so four of the six neighbours of any game are immediately identified[8].

The easiest introduction to the structure of the topological space is by identifying familiar games in the periodic table. The most famous game is the Prisoner's Dilemma. In Figure 5, the PD appears in black touching the exact centre of the table.

The symmetric games[9] include seven of the most interesting and best known 2×2 games - two versions of Coordination game, two versions of the Battle of the Sexes, Chicken, Stag Hunt, and the Prisoners's Dilemma, as well as five uninteresting games. In Figure 5 the symmetric games lie on the positive diagonal.

A related group of games have symmetric order graphs but are not symmetric in the game-theoretic sense. These *quasi-symmetric* games[10] make a series of negative diagonals.

When the periodic table is rolled into a torus it is clear there are only two quasi-symmetric diagonals, each with 12 games[11]. There is an additional 12 game diagonal parallel to the symmetric games.

[8]The other linkages may be thought of as passing through wormholes that connect the quadrants of the periodic table. In the permutation toplogy all links are equivalent but the links highlighted in the Periodic Table are those that are strongest from a game theoretic point of view.

[9]Symmetric games are games in which players face identical choices and payoffs. In terms of transformations, symmetric games are invariant under an exchange of payoffs for each outcome followed by an exchange of inducement correspondences between players.

[10]Quasi-symmetric games are invariant under an exchange of payoffs for each outcome.

[11]For readers familiar with Rapoport and Guyer's treatment, the 24 games include 12 pairs of reflections. In our approach these reflections are distinct games.

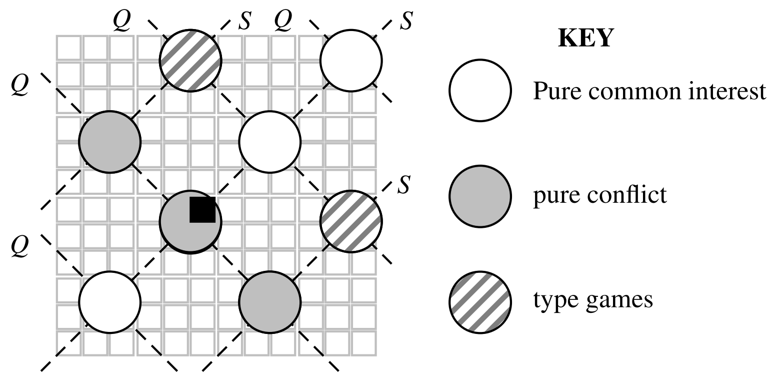


Figure 6. The pure groups and the symmetry axes

V. THE PURE CASES

Altogether, there are four diagonal axes that can best be understood as circles on the torus that intersect at eight points. These diagonal features, shown in Figure 6 as dotted lines, are intimately connected with the distribution of common and opposed interests.

The diagonals are Villarceau circles. Exactly four topologically distinct circles can be drawn through any point on a torus[24]. One is in the plane of the torus and passes around the hole like a bead of icing on the top of a donut. The second is perpendicular to the first and can be pictured as a piece of string tied through the hole. Two more, the Villarceau circles[12], run on the diagonal, winding around the donut and through the hole. Villarceau circles through a given point intersect twice.

The Villarceau circles on the axes of symmetry and quasi-symmetry intersect at eight places indicated by circles in Figure 6. These circles mark clusters of four games that are pure cases of common interest or conflict. There are three types

1. Grey circles mark three clusters of games with four negatively sloped inducement correspondences. These are the **pure conflict games**. The three clusters are joined by a band of constant-rank-sum games lying along a quasi-symmetric axis. Figure 7 shows order graphs for the 14 pure conflict games. The group includes the Prisoners's Dilemma and eight games with no pure strategy Nash equilibrium.

2. White circles mark three clusters of games with four positively sloped inducement correspondences. These are the **pure common interest** games. They fall along the symmetric diagonal. There are two pure common interest games games adjacent on a quasi-symmetric axis to one of the clusters. Figure 8 shows the order graphs for the the 14 games in this group. The group includes Coordination games and Battle of the Sexes games.

[12] For a demonstration that they really are circles, see Weisstein [24].

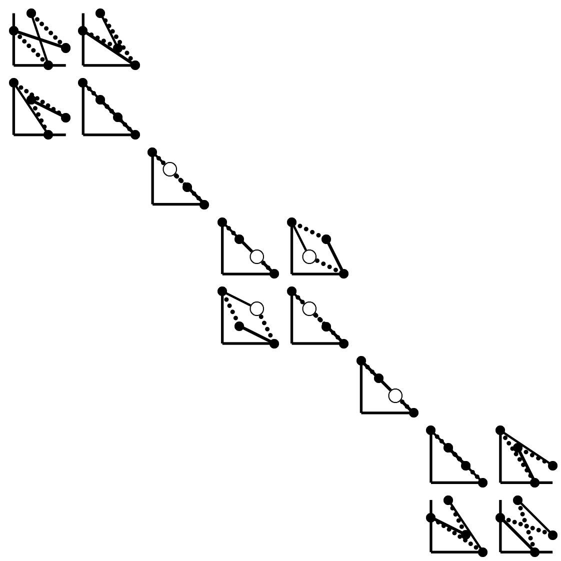


Figure 7. The pure conflict games

3. Grey and white striped circles in Figure 6 indicate games which are **pure conflict** for one player and **pure common interest** for the other. This is a new class of game that we call the "Type" games because the players are of different unmixed types. Figure 9 shows order graphs for one of the two clusters of Type games. (The other cluster is a mirror-reflection of the one shown.)

VI. MIXED MOTIVE GAMES

Figure 10 shows all games shaded according to whether they have $0, 1, 2, 3$, or 4 negatively sloped inducement correspondences. Pure conflict games are black and pure common interest games are white. The rest of the games are mixed motive games in Schelling's sense[18].

The mixed motive region includes, among others, Chicken, Stag Hunt, four games with no equilibrium and the Alibi games described in Robinson and Goforth(2004)[15]. Hatched bands around the pure conflict and pure common interest games show where games have one positively or one negatively sloped inducement correspondence. The dark and light gray regions consist of games with two positively sloped inducement correspondences and two that are negatively sloped. In the light gray games, each player has one positively sloped and one negatively sloped inducement correspondence. In other words, they both have mixed interests. In the dark gray games, one player has only positively sloped inducement correspondences and the other has only negative slopes. These

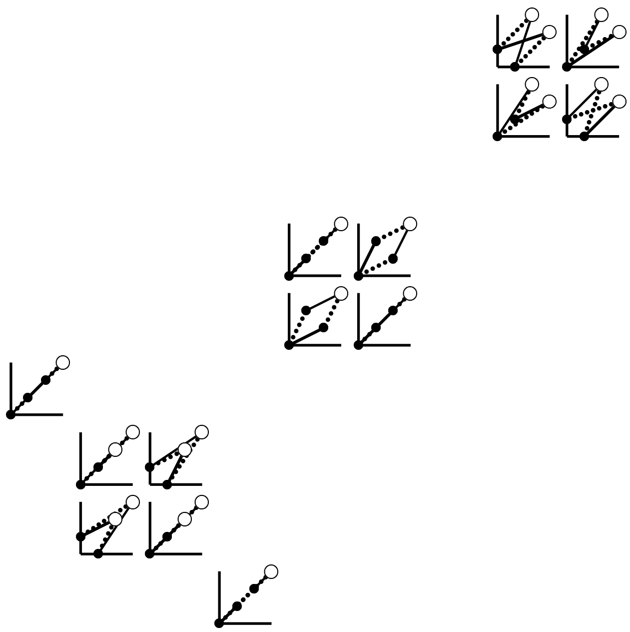


Figure 8. The pure common interest games

are the Type games.

VII. GIVER AND TAKER: THE TYPE GAMES

The two intersections marked with striped circles in Figure 6 are occupied by Type games. At each intersection there are four in a cluster with two more adjacent along the quasi-diagonal, twelve Type games in all[13].

The Type games for one of the intersections are shown in Figure 9. The players are asymmetric in the strongest way. One player always gains by making the other better off; the other always loses by making her partner better off. Unlike Chicken and Stag Hunt, in which both players have mixed interests, the motives of players in these Type games are absolutely unmixed.

In Type games the players live in different moral universes. One type is never led into temptation, the other is never free of temptation. One needs no moral instruction, the other must be restrained by law. One freely casts his bread upon the water and the Lord provides, while the other must live by theft. These games seem perfectly suited for exploring a whole class of morally ambiguous situations - cases in which agents may debate morality from fundamentally different material situations.

Isolated examples of Type games have appeared in the literature from time to time. Schelling [18], for example, used the game marked *A* in Figure 9 to describe the situation

[13]The adjacency of one game to the cluster depends on edges not shown in the periodic table.

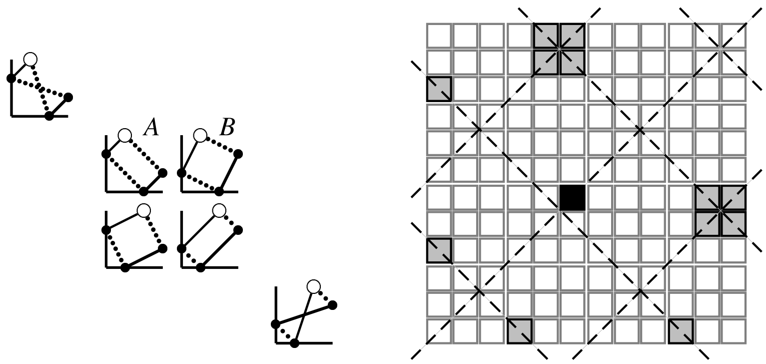


Figure 9. Pure Type games

of a blackmailer and his victim [14]. Licht [7] uses two examples to describe situations in which two nations with very different powers negotiate. His *Ideological Hegemony 1* and *2* are reassigned versions of Figure 9 *A* and *B*.

VIII. Counting conflict correctly

The number of positively and negatively sloped inducement correspondences is a useful measure of the degree of conflict or common interest in the ordinal 2×2 games. It provides a complete classification. It has significant advantages over the model proposed by Schelling in 1958 [19] because it allows us to distinguish between mixed motive games and the Type games.

Figure 11 shows how Schelling's proposal can be extended to include different types of mixed-motive ordinal games. In Figure 11 there are five degrees of alignment of interests, and three degrees of mixed motives. Along the upper arc there are three cases in which the players are completely similar in the likelihood that they will face negatively or positively sloped inducement correspondences. Below the arc are cases in which players' incentives differ. Type games on the lowest path exhibit complete differentiation.

IX. A needless extension uncovers an elegant pattern

An approach that is more consistent with the topological structure distinguishes row and column players. So far we have treated the case in which Row has two negatively sloped inducement correspondences and Column one as equivalent to a case in which Column has two and Row has one. We have implicitly assumed that players are indistinguishable, which is the assumption that Rapoport and Guyer [12] used to reduce the number of

[14] Interestingly he imagines the blackmailer unilaterally agreeing to compensate the victim in a certain situation.

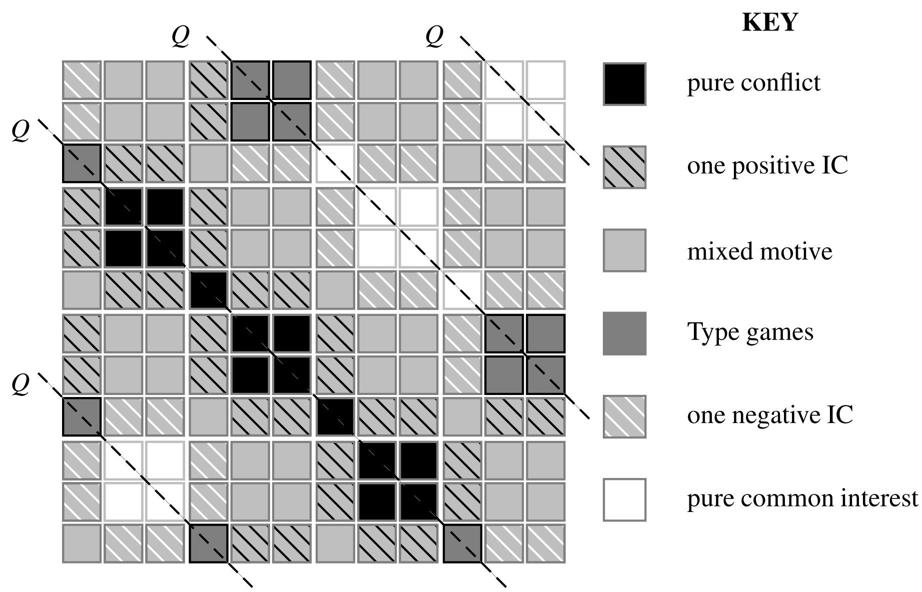


Figure 10. The Periodic table by number of negatively sloped inducement correspondences

games from 144 to 78. If players are not interchangeable there are three degrees of common interest for each player, making a total of nine possible cases.

In Figure 12, pure common interest games for Row (both her inducement correspondences are positively sloped) appear as white squares in the lower left figure. Pure conflict games for Row appear as black squares and games in which Row has mixed motives are grey. Column's motives are shown with the same colors on the upper left figure. Only half of each square is coloured for Column.

When the pattern for Column is laid over the pattern for Row the result is the remakable pattern in the centre of Figure 12. Pure conflict games are all black, and pure common interest games are entirely white. In the grey games both players have mixed motives. Black and white games are games in which players have different kinds of motivation: the Type games.

All the games with black or white - where individual players have unmixed motives - cluster around the intersections of the Villarceau circles marking the symmetric and quasi-symmetric games.

The overall pattern is even more striking if multiple copies of Figure 12 are laid edge to edge, producing the image on the cover of the paper. Tiling the plane in this way brings out features that wind around the torus. The image can be read like a topological map. In the toroidal world of the 2×2 games there are black highlands where conflict prevails and white lowlands where cooperation is the rule.

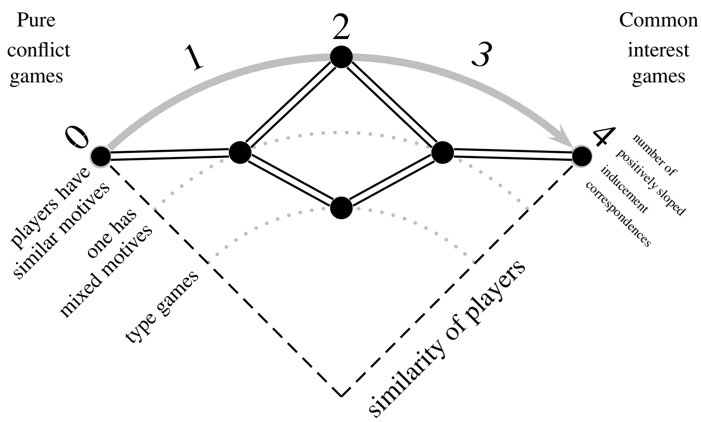


Figure 11. A discretely corrected version of Schelling's classification scheme

X. Structure of conflict

We have carried forward a project begun by Schelling in 1958. We provide a complete classification of the ordinal 2×2 games in terms of the degree of conflict among the players. We have shown that the 1966 taxonomy developed by Anatol Rapoport and Melvin Guyer provides an inconsistent and misleading treatment of conflict in the 2×2 games.

Topological relationships and symmetry allow us to uncover unexpected relationships among games of pure conflict and games of pure common interest. The fact that the pure conflict and common interest games exhibit such a clear pattern demonstrates the organizing power of the topological approach. Our treatment of the mixed motive games reveals a class of games, the Type games, that has not previously been recognized as distinct. In Type games the players live in different moral universes. These are games that will surely interest moral philosophers.

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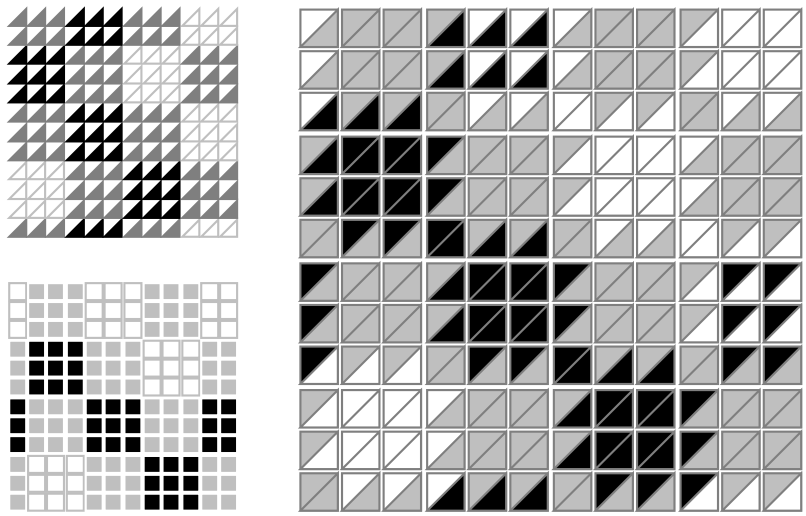

Figure 12. The Periodic table by number of negatively sloped inducement correspondences (black: 2, grey: 1, white: 0), distinguishing Row and Column players

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