The technological effect in cost sharing problems*

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Abstract

Cost sharing problems usually assume that all agents have access to a commonly owned technology and that gains from cooperation are a matter of scale. However, in many real-life problems agents have private technologies and at least part of the gain from cooperation come from the fact they when a coalition grows, the technology that they have access to improves. A simple model is built, where economies of scale are eliminated in order to study this effect. We use as the key axiom the property that if an agent never improves the technology of the coalition he joins, he should not get any part of this gain. With simple properties of linearity and symmetry, this axiom characterizes a well-defined set of rules. Adding a property of monotonicity, on technology or the addition of technologically negligible agents, or a property on the upper limit of individual allocations, we obtain a unique rule, derived from the familiar Shapley value.

Keywords: Cost sharing, technology, dummy property, Shapley value.

1 Introduction

In cost sharing problems, agents cooperate and combine their demands, to which is associated a cost. The central planner's main task is then to divide that cost among the participating agents.

Cost sharing problems usually have in common the following characteristics: all agents have access to a commonly owned technology, which associates to any level of demand a cost. The gains (or losses) associated to cooperation come from the returns to scale exhibited by the technology (or returns to scope in the case of heterogeneous goods). However, we argue that gains from cooperation can come from another important source, through technological cooperation.

Suppose that agents have privately owned technologies of production. When two agents cooperate, they not only put their demands together, but also their

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technologies. A possibly complex process, unobserved by the planner, aggregates these technologies to form a new efficient technology for the coalition. In particular, technologies can be complements, in which case the technology improves for both agents. The planner observes this technology, and the associated cost.

Consider the following two-player example:

Example 1 *Agents demand a homogeneous good. Demands by agents 1 and 2 are respectively 16 and 9 units. Costs for agent 1 alone is 240, for agent 2 alone it is 180. Suppose that when they cooperate, the total cost is 300.*

Case 1: *Technology is publicly owned, and such that the cost for a coalition S is $60\sqrt{x(S)}$, where $x(S)$ is the total quantity demanded by the coalition.*

Case 2: *Technology is privately owned, and offers constant returns to scale. The technology of agent 1 is such that his average (and marginal) cost is 15, while for agent 2 it is 20. Their technologies are complements and when they cooperate, average cost falls to 12.*

Both cases generate the same coalitional costs, but for distinct reasons. In case 1, all the gains come from economies of scale. In case 2, there are no economies of scale, but the gains come from the improvement in technology. How to allocate the gain generated by this technological effect is not trivial. In particular, we will probably not want to share it like a gain coming from returns to scale.

In practice, technology plays an important part in many situations of cooperation. Research joint ventures are the obvious examples, where cooperation can improve efficiency of the research even without increasing investments (La Manna, 2006). One can also think of countries cooperating in a space agency, or different departments of a company not only putting together their requests for a project, but also some of the workers and tools they have at their disposal to complete the task.

It is also important to note that technology is meant in the broadest sense. For example, in a network problem where the agents consist of the nodes in the network, and where cooperating agents can only use links between their nodes, a broader coalition has access to a larger set of possible links. This, in turn, is a factor that generates gains from cooperation and that has nothing to do with demands. In problems where agents have to be connected to a single source to get a good, and where cost on each link depends on the volume of the flow on it, this effect is sufficient to generate stability of the underlying game when there is decreasing returns to scale on each link (Trudeau, 2006).

The assumption that agents have private technologies has been used in the output-sharing literature, where agents put together their inputs and where we must allocate the output among the participants. Leroux (2006) studies a problem where agents have different technologies that they combine to enhance productivity, where he looks at group strategyproof mechanisms.

When the technology-improving process is well-known and depends on some inputs, and if the problem is more production-oriented, the output-sharing model is more natural. However, the cost sharing model can be used on a larger set of problems and can accommodate cases where little is known of how technology improves. For example, problems where some otherwise identical agents are exogenously able to cooperate efficiently with some and unable with others can be modelled using the proposed framework.

To better separate the so-called technology and scale effects, a simple model with constant returns to scale is used, so as to focus on the effects of technology.

To help introduce the key axiom used, consider this second example:

Example 2 *Demands are the same as in Example 1 and the privately owned technologies still give an average cost of 15 to agent 1 and 20 to agent 2. When they cooperate, however, the average cost is 15. If we apply the Shapley value to the resulting stand-alone game, it gives allocations of 217.5 to agent 1 and 157.5 to agent 2.*

The cooperation in example 2 generates a gain of 45, and with the Shapley value both agents get an equal share of this gain. However, in this case, agent 2 does not contribute anything technologically. When both agents cooperate, the efficient way to produce is simply to discard agent 2's technology. In this case, it is hard to argue that both agents should get an equal share of this technological gain.

In fact, if agents are responsible for their technologies, having possibly invested in them, we will want to reward agents with good technologies. In example 2, we can argue that no part of the technological gain should be allocated to agent 2. More generally, if an agent can be added to any (non-empty) coalition without ever improving the technology of that coalition, then he should be allocated no part of the technological gains and should pay his stand-alone cost. This will be the central axiom used in the paper, that we will call the *Technological dummy property.*

This property is a close relative of the familiar Dummy property in the classic cost sharing literature, that says that if an agent can be added to any coalition at no extra cost, than this agent should not pay anything. This property was introduced in the original paper by Shapley (1953) where it was the main equity postulate. It conveys the simple idea that agents should not pay for costs for which they are not responsible. They are therefore fully responsible for the costs of their demands. The property has been adapted to models with continuous demands. In that context, it implies that if there is an agent such that adding one unit of his good to any demand profile can be done at no extra cost, than this agent should not pay anything. See Moulin and Sprumont (2006) for a discussion of the implications of the property and a comparison to properties yielding partial responsibility.

The technological dummy property differs in the sense that we look at variations of average cost, and not total cost, when an agent joins a coalition. This coalition also needs to be non-empty, which is not the case in the classic property. We differ to section 5 further discussion on this point.

Conceptually, the property is even closer to the stronger property, for the classic model with continuous demands, that if there is an agent such that the extra cost of adding one unit of his good to any demand profile is always equal to a, than this agent should pay a times his demand. In that case, there is never any gains related to scale when the agent joins a coalition, and the property imposes that he gets none of it. This property does not give anything new in the model with publicly-owned technology and continuous demands as it is implied by the usual properties of Additivity, Non-negativity and the classic Dummy property. The cost sharing rules satisfying these properties have been characterized by Friedman (2004) and Haimanko (2000). See Wang (1999) for the case with indivisible demands and Weber (1988) for the case of the stand-alone game.

Combined with adaptations of classic properties of linearity (on both demands and technologies) and symmetry, the technological dummy property generates a well-defined class of cost sharing rules. We then define properties of monotonicity in the technology, a population monotonicity property when the added agent does not contribute anything technologically, and the classic stand-alone property. It turns that any one of these three properties, in combination with the four mentioned before, is sufficient to characterize a unique rule.

This rule is such that for all agents i, we define a game that allocates to each coalition that does not contain i the demand of agent i times the change in average cost when the coalition joins agent i. The Shapley value is then applied on this game. An agent is allocated his stand-alone cost to which we add the sum of his Shapley values on the different games.

The paper is organized as follows: Section 2 defines the framework, the proposed rule and the central axioms used. Section 3 presents the main characterization results. In Section 4, we extend the framework to variable population problems and introduce a new characterization of the proposed rule. Section 5 discusses a possible weakening of the key axiom of Technological Dummy and extensions to more general problems. Independence of axioms is proven in appendix.

2 Notations and definitions

2.1 The model

Let $N = \{1, ..., n\}$ be the (fixed) set of agents. Let $\mathcal{N} = \{S \mid \varnothing \neq S \subseteq N\}$ be the set of non-empty subsets of N, i.e., the set of possible, non-empty coalitions. Agents have demands for a single, common good. Let $x \in \mathbb{R}^n_+$ be the demand vector. Demands are supposed inelastic. If $S \in \mathcal{N}$ and $x \in \mathbb{R}^n_+$, write $x(S) := \sum_{i \in S} x_i$ and $\bar{x}(S) := \frac{x(S)}{|S|}$.

We use the simple assumption that there are constant returns to scale, that is that at a given technology, marginal cost is constant. Let $\Gamma = \left\{c \in \mathbb{R}^{\mathcal{N}}_+ \mid S \subseteq T \Rightarrow c(S) \geq c(T)\right\}$. $c \in \Gamma$ represents the average (or marginal) cost for each coalition, with the assumption that this cost cannot increase when we add agents to a coalition.

A problem is a vector $(c, x) \in \Gamma \times \mathbb{R}^n_+$, that is the cost and demand vectors.

The cost for a coalition S is simply defined as $c(S)x(S)$.

A rule is a map $y : \Gamma \times \mathbb{R}^n_+ \to \mathbb{R}^n$ s.t. $\sum_{i\in N} y_i(c,x) = c(N)x(N)$. Note that since $c(S)x(S)$ can decrease when we add agents to coalition S, we do not impose that $y_i \geq 0$. There is ample justification in this setting to subsidize an agent if he allows other agents to significantly improve on their technologies.

2.2 A proposed rule

Fix a problem (c,x). For every $i \in N$, define a function $v^i(\cdot,c)$ as follow:

$$v^i(S,c) = c(\{S \cup i\}) - c(\{i\}) \text{ for each } S \subseteq N \backslash \{i\}$$

The function $v^i(\cdot,c)$ is a TU-game on the player set $N\backslash\{i\}$. Given the problem (c,x), it gives, for each coalition, the change in the marginal cost when a coalition joins agent i. Notice that since $c(S) \geq c(T)$ for all $S \subseteq T \subseteq N$, we have that $v^i(S,c) \geq v^i(T,c)$ for any $S \subseteq T \subseteq N\backslash\{i\}$ and $v^i(S,c) \leq 0$ for all $S \subseteq N\backslash\{i\}$, all $i \in N$ and all $c \in \Gamma$.

The allocation for the problem (c,x) is then $\varphi_i(c,x) = c(\{i\})x_i + \sum_{j\in N\backslash\{i\}} x_j Sh_i(v^j(\cdot,c))$, where $Sh(v^j(\cdot,c))$ is the Shapley value on the game $v^j(\cdot,c)$.

Thus, the rule separates the cost of each player. The cost of giving agent i is demand is $c(N)x_i$, and agent i is allocated $c(\{i\})x_i$. Players in $N\backslash\{i\}$ must be allocated $(c(N) - c(\{i\}))\, x_i$. To do so, we use the Shapley value on v^i.

2.3 Properties

The following properties are central to the results in the next section.

The first two properties are along the lines of the classic Additivity property. In our context, particularly without the assumption of non-negativity of the cost-shares, we need the stronger property that the rule be linear in both the demand and the cost vectors. Because of the assumption of constant returns to scale, the Demand Linearity property is not particularly strong.

Demand linearity: For $x^1, x^2 \in \mathbb{R}^n_+$, $c \in \Gamma$ and λ_1, $\lambda_2 \in \mathbb{R}_+$, $y(c, \lambda_1 x^1 + \lambda_2 x^2) = \lambda_1 y(c,x^1) + \lambda_2 y(c,x^2)$.

Technological linearity: For c^1, $c^2 \in \Gamma$, $\beta_1, \beta_2 \in \mathbb{R}$, such that $\beta_1 c^1 + \beta_2 c^2 \in \Gamma$, $y(\beta_1 c^1 + \beta_2 c^2, x) = \beta_1 y(c^1,x) + \beta_2 y(c^2,x)$ for any $x \in \mathbb{R}^n_+$.

We also use a weak form of symmetry, where if two agents are completely identical, both in their demands and how they influence the technology, we impose that they are allocated the same amount.

Equal shares for equivalent agents: For any problem (c,x) and any $i,j \in N$, if $c(S\cup\{i\}) = c(S\cup\{j\})$ for all $S \subseteq N\backslash\{i,j\}$ and $x_i = x_j$, then $y_i(c,x) = y_j(c,x)$.

Finally, to introduce the key axiom discussed in the introduction, we need the following definition:

Definition For any $c \in \Gamma$, if $c(S \cup \{i\}) = c(S)$ for all $\varnothing \neq S \subseteq N\backslash\{i\}$, we say that i is a **technological dummy in c.**

Technological dummy property: For any problem (c,x), if $i \in N$ is a technological dummy in c, then $y_i(c,x) = c(\{i\})x_i$.

3 Results

First, remark that on any problem with only two agents, the Technological Dummy property and Technological Linearity are sufficient to characterize φ. To see this, remark that any problem $c \in \Gamma$ can be written as the sum of two technological vectors c^1 and c^2 such that one agent is a technological dummy in c^1 and the other agent is a technological dummy in c^2. There are many ways to do this, one simple way being to define c^1 such that $c^1(\{1\}) = c(\{1\}) - c(\{1,2\})$ and $c^1(\{2\}) = c^1(\{1,2\}) = 0$. We then define $c^2 = c - c^1$. The Technological Dummy property, combined with the budget-balance condition, defines a unique rule on c^1 and c^2. By Technological Linearity (in fact a weaker property of additivity is sufficient), we have also defined a unique rule on c.

On problems with larger sets of agents, the budget-balance condition together with the Technological Dummy property will not yield a unique rule, so more properties will be needed.

Before moving on to the main results, we need to introduce some notations.

Notation 1 *For each $T \in \mathcal{N} \backslash N$, define $\gamma_T \in \Gamma$ by*

$$\gamma_T(S) = \begin{cases} 0 \text{ if } S \cup T = N \\ 1 \text{ otherwise} \end{cases}$$

Moreover, define $\gamma_N \in \Gamma$ by $\gamma_N(S) = 1$ for all $S \in \mathcal{N}$.

Let $\Gamma_0 = \{\gamma_T \mid T \in \mathcal{N}\}$. By definition[1], $\Gamma_0 \subset \Gamma$. Note that γ_T is such that all $i \in T$ are technological dummies in γ_T, while agents in $N \backslash T$ are not.

We are now ready to introduce the family of rules obtained by adding properties of Demand Linearity and Equal Shares for Equivalent Agents.

Lemma 1 *If a rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity and Equal Shares for Equivalent Agents then there exists a list of $n-2$ real numbers $\alpha = \left(\alpha^1, ..., \alpha^{n-2}\right)$ such that for all $T \in \mathcal{N}$, $x \in \mathbb{R}^n_+$ and $i \in N$, $y_i(\gamma_T, x)$ is equal to*

$$y_i^\alpha(\gamma_T, x) = \left\{ \begin{array}{ll} x_i & \text{if } i \in T \\ \alpha^{|T|}\left(x_i - \bar{x}(N \backslash (T \cup \{i\}))\right) - \frac{x(T)}{n-|T|} & \text{if } i \notin T \end{array} \right\} \tag{1}$$

with the convention that $\bar{x}(\varnothing) = 0$ and $\alpha^{n-1} = 0$. Each value of α determines a unique rule on $\Gamma \times \mathbb{R}^n_+$.

Proof. The proof is in two steps. First, we will show that Γ_0 is a base of $\mathbb{R}^{\mathcal{N}}$. In the second step, we will determine cost allocations on problems $(\gamma_T, x) \in \Gamma_0 \times \mathbb{R}^n_+$ using the properties of Technological Dummy, Demand Linearity and Equal Shares for Equivalent Agents. Technological Linearity then allows us to compute cost allocations on any $(c, x) \in \Gamma \times \mathbb{R}^n_+$.

Step 1: Show that Γ_0 is a base of $\mathbb{R}^{\mathcal{N}}$

[1] Throughout this paper, $\subset$ denotes strict inclusion, i.e. $\subsetneq$.

We show that Γ_0 is a generating set of $\mathbb{R}^{\mathcal{N}}$. Since $|\Gamma_0| = |\mathcal{N}|$, Γ_0 is a base of $\mathbb{R}^{\mathcal{N}}$.

For all $S \in \mathcal{N}$, define $\mu_S \in \mathbb{R}_+^{\mathcal{N}}$ such that, for all $S \in \mathcal{N}$, $\mu_S(T) = 1$ if $S \subseteq T$ and 0 otherwise. The set $\mu = \{\mu_S\}_{S \in \mathcal{N}}$ is often used in the cost sharing literature, and it is well-known that it is a generating set. We will show that every game in μ is a linear combination of games in Γ_0.

First, for all $S \in \mathcal{N} \backslash N$, $\mu_S = \gamma_N - \gamma_{N \backslash S}$, as, by definition of γ_N and $\gamma_{N \backslash S}$, $\mu_S(T) = 1$ if $S \subseteq T$ and 0 otherwise.

Define

$$\mu_N = \gamma_N - \sum_{S \in \mathcal{N} \backslash N} (-1)^{|S|+1} \gamma_S$$

We check that for $T \in \mathcal{N} \backslash N, \mu_N(T) = 0$ while $\mu_N(N) = 1$.

$$\begin{aligned}
\mu_N(T) &= \gamma_N(T) - \sum_{S \in \mathcal{N} \backslash N} (-1)^{|S|+1} \gamma_S(T) \\
&= 1 - \sum_{\substack{S \in \mathcal{N} \backslash N \\ S \cup T \neq N}} (-1)^{|S|+1} \\
&= 1 - \sum_{S \in \mathcal{N} \backslash N} (-1)^{|S|+1} + \sum_{\substack{S \in \mathcal{N} \backslash N \\ S \cup T = N}} (-1)^{|S|+1} \\
&= 1 - \sum_{\substack{S \subset N \\ S \neq \varnothing}} (-1)^{|S|+1} + \sum_{R \subset T} (-1)^{n-|T|+|R|+1} \\
&= 1 + (-1)^1 + (-1)^{n-1} - (-1)^{n-1} \\
&= 0
\end{aligned}$$

Since $\gamma_S(N) = 0$ for all $S \in \mathcal{N} \backslash N$ and $\gamma_N(N) = 1$, $\mu_N(N) = 1$.

As claimed, every game in μ can be written as a linear combination of games in Γ_0.

Since Γ_0 is a generating set and $|\Gamma_0| = |\mathcal{N}|$, every $c \in \Gamma$ can be written as a unique linear combination of the elements of Γ_0.

Step 2: Show that for all $i \in N$ and $x \in \mathbb{R}_+^n$ shares on the base problems $(\gamma_T, x) \in \Gamma_0 \times \mathbb{R}_+^n$ can be written as in (1).

This will be done in three substeps. First, we find the allocation $y_i(\gamma_T, x)$ when $i \in T$, which, with budget balance, allows us to determine all cost allocations when $|T| \geq n-1$. Next, we find the allocation $y_i(\gamma_T, x)$ when $i \notin T$ and show that it depends on a parameter α^T. Finally, we show that $|S| = |T| \Rightarrow \alpha^T = \alpha^S$.

Step 2.1: Show that $y_i(\gamma_T, x) = x_i$ for all $i \in T$, $T \in \mathcal{N}$, $x \in \mathbb{R}_+^n$.

For any $T \in \mathcal{N}$, we have that $\gamma_T(S \cup \{i\}) = \gamma_T(S)$ for every $\varnothing \neq S \subseteq N \backslash \{i\}$ and every $i \in T$, that is each $i \in T$ is a technological dummy in γ_T. By the Technological Dummy Property, $y_i(\gamma_T, x) = x_i$ for all $i \in T$ and all $T \in \mathcal{N}$.

Combined with budget balance, it implies that for any $i \in N$, $y_i(\gamma_{N \backslash \{i\}}, x) = -x(N \backslash \{i\})$, since $\gamma_{N \backslash \{i\}}(N) = 0$.

We have thus defined unique allocations on $(\gamma_{N\backslash\{i\}}, x)$ and (γ_N, x).

Step 2.2: Show that, for every $T \in \mathcal{N}$ such that $|T| < n-1$, we have

$$y_i(\gamma_T, x) = \alpha^T \left(x_i - \bar{x}(N\backslash(T \cup \{i\}))\right) - \frac{x(T)}{n-|T|} \text{ if } i \notin T$$

with $\alpha^T \in \mathbb{R}$.

Fix $T \in \mathcal{N}$ such that $|T| < n-1$ and $i, j \in N\backslash T$ until the end of step 2.2.

Since $\gamma_T(N) = 0$, Step 2.1 and budget balance imply that

$$\sum_{k \in N\backslash T} y_k(\gamma_T, x) = -x(T) \quad \text{(a)}$$

For any $S \in \mathcal{N}$, define $\mathbf{1}^S \in \mathbb{R}^n_+$ as follows: $\mathbf{1}^S_k = 1$ if $k \in S$ and $\mathbf{1}^S_k = 0$ otherwise. In particular, $\mathbf{1}^k$ is such that agent k demands 1 while everybody else has demand 0. Any vector x can be rewritten as $x = \sum_{k\in N} x_k \mathbf{1}^k$. Let $\alpha^T_{kl} := y_k(\gamma_T, \mathbf{1}^l)$ for $k, l \in N$ be the cost allocation of agent k in the problem $(\gamma_T, \mathbf{1}^l)$. By Step 2.1 we have,

$$\text{if } k \in T, \alpha^T_{kl} = 0 \text{ for all } l \neq k \text{ and } \alpha^T_{kk} = 1 \quad \text{(b)}$$

Then, we have

$$\begin{aligned} y_i(\gamma_T, x) &= y_i(\gamma_T, \sum_{k\in N} x_k \mathbf{1}^k) \\ &= \sum_{k \in N} x_k y_i(\gamma_T, \mathbf{1}^k) \text{ (By Demand Linearity)} \\ &= \sum_{k\in N} \alpha^T_{ik} x_k \end{aligned}$$

From (a) and (b)

$$\sum_{k\in N\backslash T} \alpha^T_{kl} = -1 \text{ for } l \in T \quad (2)$$

$$\sum_{k\in N\backslash T} \alpha^T_{kl} = 0 \text{ for } l \in N\backslash T \quad (3)$$

Since $i, j \in N\backslash T$, we have that $\gamma_T(S \cup \{i\}) = \gamma_T(S \cup \{j\})$ for all $S \in N\backslash\{i,j\}$. If $x_i = x_j$, then by Equal Shares of Equivalent Agents, $y_i(\gamma_T, x) = y_j(\gamma_T, x)$.

Consider the demand vector $\mathbf{1}^k$, for $k \in T$. Since $\mathbf{1}^k_i = \mathbf{1}^k_j = 0$, by Equal Shares for Equivalent Agents we must have that $y_i(\gamma_T, \mathbf{1}^k) = \alpha^T_{ik} = \alpha^T_{jk} = y_j(\gamma_T, \mathbf{1}^k)$. Combined with (2), we have that $\alpha^T_{ik} = -\frac{1}{n-|T|}$ for all $k \in T$.

Case 1: $|T| = n-2$. Consider the demand vector $\mathbf{1}^{\{i,j\}}$. By Equal Shares for Equivalent Agents, $y_i(\gamma_T, \mathbf{1}^{\{i,j\}}) = \alpha^T_{ii} + \alpha^T_{ij} = \alpha^T_{ji} + \alpha^T_{jj} = y_j(\gamma_T, \mathbf{1}^{\{i,j\}})$.

By (3), $\alpha_{ij}^T = -\alpha_{jj}^T$ and $\alpha_{ji}^T = -\alpha_{ii}^T$. Therefore, $y_i(\gamma_T, \mathbf{1}^{\{i,j\}}) = \alpha_{ii}^T - \alpha_{jj}^T = -\alpha_{ii}^T + \alpha_{jj}^T = y_j(\gamma_T, \mathbf{1}^{\{i,j\}})$ and $\alpha_{ii}^T = \alpha_{jj}^T$. Define $\alpha^T = \alpha_{jj}^T$. For any $x \in \mathbb{R}_+^n$, $y_i(\gamma_T, x) = \alpha^T(x_i - x_j) - \frac{x(T)}{2}$.

Case 2: $|T| < n-2$. Consider the demand vector $\mathbf{1}^k$ for $k \in N\backslash(T \cup \{i,j\})$. Since $\mathbf{1}_i^k = \mathbf{1}_j^k = 0$, by Equal Shares for Equivalent Agents we must have that $y_i(\gamma_T, \mathbf{1}^k) = \alpha_{ik}^T = \alpha_{jk}^T = y_j(\gamma_T, \mathbf{1}^k)$. Combined with (3), we have that

$$\alpha_{ik}^T = -\frac{\alpha_{kk}^T}{n - |T| - 1} \tag{4}$$

for all $k \in N\backslash T$.

Consider the demand vector $\mathbf{1}^{\{i,j\}}$. By Equal Shares for Equivalent Agents, $y_i(\gamma_T, \mathbf{1}^{\{i,j\}}) = \alpha_{ii}^T + \alpha_{ij}^T = \alpha_{ji}^T + \alpha_{jj}^T = y_j(\gamma_T, \mathbf{1}^{\{i,j\}})$. By (4), it implies that $\alpha_{ii}^T - \frac{\alpha_{jj}^T}{n-|T|-1} = -\frac{\alpha_{ii}^T}{n-|T|-1} + \alpha_{jj}^T$. After simplification, we get that $\alpha_{ii}^T = \alpha_{jj}^T$. Define $\alpha^T = \alpha_{jj}^T$.

As claimed, we have that for every $T \in \mathcal{N}$ such that $|T| < n-1$ and every $k \notin T$

$$\begin{aligned} y_k(\gamma_T, x) &= \alpha^T \left(x_k - \sum_{l \in N\backslash(T\cup\{k\})} \frac{x_l}{n-|T|-1} \right) - \sum_{l \in T} \frac{x_l}{n-|T|} \\ &= \alpha^T \left(x_k - \bar{x}(N\backslash(T \cup \{k\})) \right) - \frac{x(T)}{n-|T|} \end{aligned} \tag{5}$$

Step 2.3: Show that for any $S, T \in \mathcal{N}$ such that $|S| = |T|$, then $\alpha^S = \alpha^T$

Fix $i, j \in N$, $S \subset N\backslash\{i,j\}$ and $x \in \mathbb{R}_+^n$. Define $\theta = \gamma_{S\cup\{i\}} + \gamma_{S\cup\{j\}}$. Then, for $R \in \mathcal{N}$

$$\theta(R) = \begin{cases} 2 \text{ if } R \cup S \cup \{i\} \neq N \text{ and } R \cup S \cup \{j\} \neq N \\ 1 \text{ if } R \cup S \cup \{k\} = N \text{ and } R \cup S \cup \{l\} \neq N \text{ for } \{k,l\} = \{i,j\} \\ 0 \text{ if } R \cup S = N \end{cases}$$

Clearly, $\theta \in \Gamma$. We have that $\theta(R \cup \{i\}) = \theta(R \cup \{j\})$ for all $R \in N\backslash\{i,j\}$ and by Equal Shares for Equivalent Agents, if $x_i = x_j$, then $y_i(\theta, x) = y_j(\theta, x)$. By Technological Linearity, $y(\theta, x) = y(\gamma_{S\cup\{i\}}, x) + y(\gamma_{S\cup\{j\}}, x)$.

Then, by (5),

$$y_i(\theta, x) = x_i + \alpha^{S\cup\{j\}} \left(x_i - \bar{x}(N\backslash(S \cup \{i,j\})) \right) - \frac{x(S \cup \{j\})}{n - |S| - 1} =$$

$$\alpha^{S\cup\{i\}} \left(x_j - \bar{x}(N\backslash(S \cup \{i,j\})) \right) - \frac{x(S \cup \{i\})}{n - |S| - 1} + x_j = y_j(\theta, x)$$

which directly implies that $\alpha^{S\cup\{i\}} = \alpha^{S\cup\{j\}}$. This in turn implies that $\alpha^S = \alpha^T$ if $|S| = |T|$.

Summing up the results of steps 2.1, 2.2 and 2.3, we get that $y = y^\alpha$. With the result of Step 1, it implies that α defines a unique rule on $\Gamma \times \mathbb{R}_+^n$. ■

Rules in this set differ only by how they allocate costs between non-technological dummies in the basic problems, which results in different values for α. In particular, φ corresponds to $\alpha = (1, ..., 1)$.

To arrive to a characterization of φ, we need to introduce new properties. The first one is the classic stand-alone property that requires that every agent should not be allocated more than his stand-alone cost.

Stand alone property: For any $i \in N$, $c \in \Gamma$, $x \in \mathbb{R}_+^n$, $y_i(c, x) \leq c(\{i\})x_i$

We also introduce a second property, a monotonicity property, that imposes that starting from a cost vector $c \in \Gamma$, if the average cost decreases for a coalition S and stays the same for all other coalitions, then all agents in S should not see their cost share increase following this change.

Technological Monotonicity: For any $i \in N$, if $c, c' \in \Gamma$ are such that there is a $S \subseteq N\backslash\{i\}$ with $c(S \cup \{i\}) > c'(S \cup \{i\})$ and $c(T) = c'(T)$ for all $T \neq S \cup \{i\}$, then $y_i(c, x) \geq y_i(c', x)$ for any $x \in \mathbb{R}_+^n$.

With these properties, we are ready for the main characterization results.

Theorem 1 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equivalent Agents and Stand alone property if and only if $y = \varphi$.*

Proof. See the appendix for the proof that φ satisfies the five properties. We show that they imply a unique rule.

Fix $i \in N$. Fix $T \subset N\backslash\{i\}$, with $T \neq \varnothing$.

By Lemma 1, we have $y_i(\gamma_T, \mathbf{1}^i) = \alpha^{|T|}$. By definition of γ_T, $\gamma_T(\{i\}) = 1$. By the Stand-alone property, we must have

$$\alpha^{|T|} \leq 1 \tag{6}$$

Define $c^T = \gamma_{\{i\}} + \gamma_{N\backslash\{i\}} + \gamma_{T\cup\{i\}} - \gamma_T$. By definition of the elements of Γ_0, we have, for any $S \in \mathcal{N}$

$$c^T(S) = \begin{cases} 2 \text{ if } i \notin S \text{ and } S \cup T \neq N\backslash\{i\} \\ 1 \text{ if } [N\backslash\{i\} \nsubseteq S \text{ and } i \in S] \text{ or } [N\backslash\{i\} \nsubseteq S \text{ and } S \cup T = N\backslash\{i\}] \\ 0 \text{ if } N\backslash\{i\} \subseteq S \end{cases} \tag{7}$$

Therefore, $c^T(S \cup \{j\}) \leq c^T(S)$ for all $S \in \mathcal{N}$ and all $j \in N$, which imply $c^T \in \Gamma$.

By Technological Linearity, $y_i(c^T, x) = y_i(\gamma_{\{i\}}, x) + y_i(\gamma_{N\backslash\{i\}}, x) + y_i(\gamma_{T\cup\{i\}}, x) - y_i(\gamma_T, x)$. By Lemma 1, we have

$$\begin{aligned} y_i(c^T, \mathbf{1}^i) &= 1 + 0 + 1 - \alpha^{|T|} \\ &= 2 - \alpha^{|T|} \end{aligned}$$

By (7), $c^T(\{i\}) = 1$. By the Stand-alone property, we must have

$$\begin{aligned} y_i(c^T, \mathbf{1}^i) &\leq c^T(\{i\}) \\ 2 - \alpha^{|T|} &\leq 1 \\ \alpha^{|T|} &\geq 1 \end{aligned}$$

With (6), we conclude that $\alpha^{|T|} = 1$.

We thus have that $\alpha^k = 1$ for all $k \in 1, 2, ..., n-2$. By Lemma 1, we have defined a unique rule on $\Gamma \times \mathbb{R}^n_+$ and we must have that $y = \varphi$. ■

Theorem 2 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equivalent Agents and Technological Monotonicity if and only if $y = \varphi$.*

Proof. See the appendix for the proof that φ satisfies the five properties. We show that they imply a unique rule.

Fix $i \in N$. Fix $T \subset N \backslash \{i\}$, with $T \neq \varnothing$.

By definition of γ_T and γ_N, we have for all $S \subseteq N \backslash \{i\}$, $\gamma_T(S) = \gamma_N(S) = 1$. We can then construct a sequence $c^0, ..., c^k$ such that $c^0 = \gamma_T$ and $c^k = \gamma_N$ with the property that for each $l \in [1, .., k]$, $c^l \in \Gamma$ and there exists a $S \in \mathcal{N}$ such that $i \in S$, $c^{l-1}(S) > c^l(S)$ and $c^{l-1}(R) = c^l(R)$ for every $R \in \mathcal{N} \backslash S$. To do so, let $\sigma = (\sigma_1, ..., \sigma_{2^{|T|}})$ be an ordering of the elements of the set $\{S \in \mathcal{N} \mid \varnothing \neq S \subseteq N \backslash T\}$. Let $\sigma_l \in \mathcal{N}$ be the element in position l. We select an ordering such that for $l < m$, $|\sigma_l| \leq |\sigma_m|$. Since for all $\varnothing \neq S \subseteq N \backslash T$, $\gamma_T(S) = 0$, we construct the sequence $c^0, ..., c^{2^{|T|}}$ as follows: at each step $l = 1, ..., 2^{|T|}$, let c^l be such that $c^l(\sigma^l) = c^{l-1}(\sigma^l) + 1 = 1$ and $c^l(R) = c^{l-1}(R)$ for every $R \in \mathcal{N} \backslash \sigma^l$.

Successive applications of Technological Monotonicity imply that $y_i(\gamma_T, x) \leq y_i(\gamma_N, x)$. By Lemma 1, we have $y_i(\gamma_T, \mathbf{1}^i) = \alpha^{|T|}$ and $y_i(\gamma_N, \mathbf{1}^i) = 1$. Therefore,

$$\alpha^{|T|} \leq 1 \tag{8}$$

We claim that $\alpha^{|T|} \geq 1$

To prove this claim, define $\gamma^i = \gamma_{N\backslash\{i\}} + \sum_{S \subset N \backslash \{i\}} (-1)^{|S|+n} \gamma_{S \cup \{i\}}$.

For $R \in \mathcal{N}$, we have that

$$\begin{aligned}
\gamma^i(R) &= \gamma_{N\backslash\{i\}}(R) + \sum_{S \subset N\backslash\{i\}} (-1)^{|S|+n} \gamma_{S\cup\{i\}}(R) \\
&= \gamma_{N\backslash\{i\}}(R) + \sum_{\substack{S \subset N\backslash\{i\} \\ R\cup S\cup\{i\} \neq N}} (-1)^{|S|+n}
\end{aligned}$$

If $R = \{i\}$, $\{S \subset N \backslash \{i\} \mid R \cup S \cup \{i\} = N\} = \varnothing$. Therefore,

$$\begin{aligned}
\gamma^i(\{i\}) &= \gamma_{N\backslash\{i\}}(\{i\}) + \sum_{S \subset N\backslash\{i\}} (-1)^{|S|+n} \\
&= 0 + \sum_{k=0}^{n-2} (-1)^{k+n} \frac{(n-1)!}{k!(n-k-1)!} \\
&= -(-1)^{2n-1} \\
&= 1
\end{aligned}$$

If $R \in \mathcal{N} \backslash \{i\}$,

$$\begin{aligned}\gamma^i(R) &= \gamma_{N\backslash\{i\}}(R) + \sum_{S\subset N\backslash\{i\}} (-1)^{|S|+n} - \sum_{\substack{S\subset N\backslash\{i\}\\ R\cup S\cup\{i\}=N}} (-1)^{|S|+n} \\ &= \gamma_{N\backslash\{i\}}(R) + \sum_{S\subset N\backslash\{i\}} (-1)^{|S|+n} - \sum_{P\subset R} (-1)^{|P|-|R|+2n-1} \\ &= \gamma_{N\backslash\{i\}}(R) - (-1)^{2n-1} + (-1)^{2n-1} \\ &= \gamma_{N\backslash\{i\}}(R)\end{aligned}$$

Therefore, $\gamma^i(R) = 0$ if $i \in R$ and $R \in \mathcal{N} \backslash \{i\}$, 1 otherwise and $\gamma^i \in \Gamma$. By definition of γ^i and γ_T, we have, for all $S \subseteq N\backslash\{i\}$, that $\gamma^i(S \cup \{i\}) \leq \gamma_T(S \cup \{i\})$ and $\gamma^i(S) = \gamma_T(S)$. By the same argument as earlier, successive applications of Technological Monotonicity yield

$$y_i(\gamma^i, x) \leq y_i(\gamma_T, x) \tag{9}$$

By Technological Linearity, $y_i(\gamma^i, x) = y_i(\gamma_{N\backslash\{i\}}, x) + \sum_{S\subset N\backslash\{i\}} (-1)^{|S|} y_i(\gamma_{S\cup\{i\}}, x)$. By Lemma 1 and our selected x, $y_i(\gamma_{N\backslash\{i\}}, x) = 0$ and $y_i(\gamma_{S\cup\{i\}}, x) = 1$ for all $S \subset N\backslash\{i\}$. Then,

$$y_i(\gamma^i, x) = \sum_{S\subset N\backslash\{i\}} (-1)^{|S|} = 1 \tag{10}$$

By Lemma 1 and our selected x, $y_i(\gamma_T, x) = \alpha^{|T|}$.

Together with (9) and (10), this implies that $\alpha^{|T|} \geq 1$.

With (8), we conclude that $\alpha^{|T|} = 1$.

Therefore, for all $k \in 1, 2, ..., n-2$, $\alpha^k = 1$. By Lemma 1, we have defined a unique rule on $\Gamma \times \mathbb{R}^n_+$ and we must have that $y = \varphi$. ■

4 Variable population problems

We now look at problems with variable population. The model is adapted to this framework as follows:

Let $N = \{1, ..., n\}$ be the (finite) set of admissible agents. For any $T \in \mathcal{N}$, let $\mathcal{T} = \{S \mid \varnothing \neq S \subseteq T\}$ be the set of non-empty subsets of T. Let $x \in \mathbb{R}^{|T|}_+$ be the demand vector. Let $\Gamma^T = \left\{c \in \mathbb{R}^{\mathcal{T}}_+ \mid R \subseteq S \in \mathcal{T} \Rightarrow c(R) \geq c(S)\right\}$.

Our domain is $\mathcal{D} = \left\{(T, c, x) \mid T \in \mathcal{N},\ c \in \Gamma^T,\ x \in \mathbb{R}^{|T|}_+\right\}$ and a rule is a map $y : \mathcal{D} \to \cup_{S\in\mathcal{N}} \mathbb{R}^{|S|}$ s.t. $y(T, c, x) \in \mathbb{R}^{|T|}$ and $\sum_{i\in T} y_i(T, c, x) = c(T)x(T)$.

To define φ in this setting, fix a problem (T, c, x). For every $i \in T$, define a function $v^i_T(\cdot, c)$ as follow:

$$v^i_T(S, c) = c(\{S \cup i\}) - c(\{i\}) \text{ for each } S \subseteq T\backslash\{i\}$$

The function $v^i_T(\cdot, c)$ is a TU-game on the player set $T\backslash\{i\}$. The allocation for the problem (T, c, x) is then $\varphi_i(T, c, x) = c(\{i\})x_i + \sum_{j\in T\backslash\{i\}} x_j Sh_i(v^j_T(\cdot, c))$.

Notation 2 *For $S \in \mathcal{T}$, and $x \in \mathbb{R}^{|T|}$, define x^s such that $x_i^s = x_i$ for all $i \in S$.*

For each $T \in \mathcal{N}$ and $S \in \mathcal{N} \backslash N$, define $\gamma_S^T \in \Gamma^T$ by

$$\gamma_S^T(R) = \begin{cases} 0 \text{ if } R \cup S = T \\ 1 \text{ otherwise} \end{cases}$$

Moreover, define $\gamma_T^T \in \Gamma^T$ by $\gamma_T^T(S) = 1$ for all $S \in \mathcal{T}$. Let $\Gamma_0^T = \left\{\gamma_S^T \mid S \in \mathcal{T}\right\}$.

Note that all axioms defined in the previous section have direct extensions on variable population problems. The result of Lemma 1 can also be extended to this framework.

Lemma 2 *If a rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity and Equal Shares for Equivalent Agents then there exists, for each $T \in \mathcal{N}$, such that $|T| \geq 2$, a list of $|T|-2$ real numbers $\alpha(T) = \left(\alpha^1(T), ..., \alpha^{|T|-2}(T)\right)$ such that for all $S \in \mathcal{T}$, $x \in \mathbb{R}_+^{|T|}$ and $i \in T$, $y_i(T, \gamma_S^T, x)$ is equal to*

$$y_i^\alpha(T, \gamma_S^T, x) = \begin{cases} x_i & \text{if } i \in S \\ \alpha^{|S|}(T)\left(x_i - \bar{x}(T \backslash (S \cup \{i\}))\right) - \frac{x(S)}{|T|-|S|} & \text{if } i \notin S \end{cases}$$

with the convention that $\bar{x}(\varnothing) = 0$ and $\alpha^{|T|-1}(T) = 0$. Each value of $\alpha = \{\alpha(T) \mid T \in \mathcal{N},\ |T| > 2\}$ determines a unique rule on $\mathcal{D}$.

Proof. Follows directly from lemma 1.

In variable population problems, a property on the addition of technological dummies can be defined. More specifically, the following property imposes that when we add a new agent to a problem, if that agent is technologically dummy and all other costs and demands stay the same, the cost shares of any of the original agents should not increase. ■

T-dummy monotonicity: For any $T \in \mathcal{N}$, if we have, for $j \in N \backslash T$, $c \in \Gamma^T$ and $c' \in \Gamma^{T \cup \{j\}}$, that $c'(S) = c(S)$ and $c'(S) = c'(S \cup \{j\})$ for all $S \in \mathcal{T}$, then $y_i(T \cup \{j\}, c', x) \leq y_i(T, c, x^T)$ for all $i \in T$ and $x \in \mathbb{R}_+^{|T|+1}$.

Adding the T-dummy Monotonicity property to the four central axioms again allows us to characterize φ.

Theorem 3 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equivalent Agents and T-dummy Monotonicity if and only if $y = \varphi$.*

Proof. See the appendix for the proof that φ satisfies the five properties. We show that they imply a unique rule.

We use the assumption that $y(T, \cdot, \cdot) = \varphi(T, \cdot, \cdot)$ for all $T \in \mathcal{N}$ such that $|T| = k$, with $1 < k < n$ and show that it implies that $y(T', \cdot, \cdot) = \varphi(T', \cdot, \cdot)$ for all $T' \in \mathcal{N}$ such that $|T'| = k+1$.

Fix $T \in \mathcal{N} \backslash N$, $i \in T$, $\varnothing \neq S \subset T \backslash \{i\}$ and $j \in N \backslash T$. Fix $x \in \mathbb{R}_+^{|T|+1}$ such that $x_i = 1$ and $x_k = 0$ for all $k \neq i$.

Step 1: Show that $\alpha^k(T \cup \{j\}) = 1$ for $1 < k \leq |T| - 1$

By definition, γ_S^T and $\gamma_{S\cup\{j\}}^{T\cup\{j\}}$ are such that j is a technological dummy in $\gamma_{S\cup\{j\}}^{T\cup\{j\}}$ and $\gamma_S^T(R) = \gamma_{S\cup\{j\}}^{T\cup\{j\}}(R)$ for all $R \in \mathcal{T}$. By T-dummy Monotonicity, we have that $y_i(T, \gamma_S^T, x^T) \geq y_i(T \cup \{j\}, \gamma_{S\cup\{j\}}^{T\cup\{j\}}, x)$. By Lemma 2 and our selected x, this amounts to $\alpha^{|S|}(T) \geq \alpha^{|S|+1}(T \cup \{j\})$. Since, by assumption, $y(T, \cdot, \cdot) = \varphi(T, \cdot, \cdot)$, we have $\alpha^{|S|}(T) = 1$. Therefore,

$$\alpha^{|S|+1}(T \cup \{j\}) \leq 1 \tag{11}$$

Define $c^S = \gamma_{\{i\}}^T + \gamma_{T\setminus\{i\}}^T + \gamma_{S\cup\{i\}}^T - \gamma_S^T$.

By definition of the elements of Γ_0^T, we have that, for any $R \in \mathcal{T}$

$$c^S(R) = \begin{cases} 2 \text{ if } i \notin R \text{ and } R \cup S \neq T\setminus\{i\} \\ 1 \text{ if } \{T\setminus\{i\} \nsubseteq R \text{ and } i \in R\} \text{ or } \{T\setminus\{i\} \nsubseteq R \text{ and } R \cup S = T\setminus\{i\}\} \\ 0 \text{ if } T\setminus\{i\} \subseteq R \end{cases}$$

and thus $c^S \in \Gamma^T$.

Define $c^{S_j} = \gamma_{\{i,j\}}^{T\cup\{j\}} + \gamma_{T\cup\{j\}\setminus\{i\}}^{T\cup\{j\}} + \gamma_{S\cup\{i,j\}}^{T\cup\{j\}} - \gamma_{S\cup\{j\}}^{T\cup\{j\}}$.

Since c^{S_j} is built using the same technological vectors as c^S, to which we have added j as a technological dummy, $c^S(R) = c^{S_j}(R)$ for all $R \in \mathcal{T}$ and j is a technological dummy in c^{S_j}. By Technological Linearity, $y_i(T, c^S, x^T) = y_i(T, \gamma_{\{i\}}^T, x^T) + y_i(T, \gamma_{T\setminus\{i\}}^T, x^T) + y_i(T, \gamma_{S\cup\{i\}}^T, x^T) - y_i(T, \gamma_S^T, x^T)$. By Lemma 2 and our selected x, $y_i(T, c^S, x^T) = 1 + 0 + 1 - \alpha^{|S|}(T)$. Since, by assumption, $y(T, \cdot, \cdot) = \varphi(T, \cdot, \cdot)$, we have $\alpha^{|S|}(T) = 1$ and $y_i(T, c^S, x^T) = 1$. In the same manner, $y_i(T \cup \{j\}, c^{S_j}, x) = 2 - \alpha^{|S|+1}(T \cup \{j\})$. By T-dummy Monotonicity, $y_i(T, c^S, x^T) \geq y_i(T \cup \{j\}, c^{S_j}, x)$, which gives

$$\alpha^{|S|+1}(T \cup \{j\}) \geq 1$$

Together with (11), this gives that $\alpha^k(T \cup \{j\}) = 1$ for $1 < k \leq |T| - 1$

Step 2: Show that $\alpha^1(T \cup \{j\}) = 1$

Define $c_\varnothing = \sum_{R\in\mathcal{T}\setminus T}(-1)^{|R|+1}\gamma_R^T$. For all $P \in \mathcal{T}\setminus T$, we have

$$\begin{aligned} c_\varnothing(P) &= \sum_{R\in\mathcal{T}\setminus T}(-1)^{|R|+1}\gamma_R^T(P) = \sum_{\substack{R\in\mathcal{T}\setminus T \\ P\cup R\neq N}}(-1)^{|R|+1} \\ &= \sum_{R\in\mathcal{T}\setminus T}(-1)^{|R|+1} - \sum_{\substack{R\in\mathcal{T}\setminus T \\ P\cup R=N}}(-1)^{|R|+1} \\ &= \sum_{\substack{R\subset T \\ R\neq\varnothing}}(-1)^{|R|+1} - \sum_{M\subset P}(-1)^{|T|-|P|+|M|+1} \\ &= -(-1) - (-1)^{|T|+1} + (-1)^{|T|+1} = 1 \end{aligned}$$

while $c_\varnothing(T) = 0$. Clearly, $c_\varnothing \in \Gamma^T$. By Technological Linearity, $y_i(T, c_\varnothing, x^T) = \sum_{R\in\mathcal{T}\setminus T}(-1)^{|R|+1}y_i(T, \gamma_R^T, x^T)$. By Lemma 2, the assumption that $y(T, \cdot, \cdot) =$

$\varphi(T,\cdot,\cdot)$ and our selected x, $y_i(T,\gamma_R^T,x^T)=1$ for all $R\in\mathcal{T}$ such that $R\neq T\setminus\{i\}$ and $y_i(T,\gamma^T_{T\setminus\{i\}},x^T)=0$. Therefore,

$$\begin{aligned}
y_i(T,c_\varnothing,x^T) &= \sum_{R\in\mathcal{T}\setminus T}(-1)^{|R|+1}-(-1)^{|T|}\\
&= \sum_{k=1}^{|T|-1}(-1)^{k+1}\frac{|T|!}{k!(t-k)!}-(-1)^{|T|}\\
&= -(-1)-(-1)^{|T|+1}-(-1)^{|T|}\\
&= 1
\end{aligned}$$

By definition of $\gamma^{T\cup\{j\}}_{\{j\}}$, $c_\varnothing(R)=\gamma^{T\cup\{j\}}_{\{j\}}(R)$ for all $R\in\mathcal{T}$ and j is a technological dummy in $\gamma^{T\cup\{j\}}_{\{j\}}$. By T-dummy Monotonicity, $y_i(T,c_\varnothing,x^T)\geq y_i(T\cup\{j\},\gamma^{T\cup\{j\}}_{\{j\}},x)$, which implies that

$$\alpha^1(T\cup\{j\})\leq 1 \tag{12}$$

To show that $\alpha^1(T\cup\{j\})\geq 1$, we need to define two more technological vectors. Define $c^{T\setminus\{i\}}=2\gamma^T_{\{i\}}+\gamma^T_{T\setminus\{i\}}-c_\varnothing$. By definitions of $c_\varnothing$ and the elements of Γ^T_0, for any $R\in\mathcal{T}$

$$c^{T\setminus\{i\}}(R)=\begin{cases}2 \text{ if } i\notin R \text{ and } R\neq T\setminus\{i\}\\ 1 \text{ if } i\in R \text{ and } R\neq T\\ 0 \text{ if } T\setminus\{i\}\subseteq R\end{cases}$$

and $c^{T\setminus\{i\}}\in\Gamma^T$.

Define $c^j=2\gamma^{T\cup\{j\}}_{\{i,j\}}+\gamma^{T\cup\{j\}}_{T\cup\{j\}\setminus\{i\}}-\gamma^{T\cup\{j\}}_{\{j\}}$. By definition, $\gamma^T_{\{i\}}$ and $\gamma^{T\cup\{j\}}_{\{i,j\}}$ are such that $\gamma^{T\cup\{j\}}_{\{i,j\}}(R)=\gamma^T_{\{i\}}(R)$ for all $R\in\mathcal{T}$ and j is a technological dummy in $\gamma^{T\cup\{j\}}_{\{i,j\}}$. The same is true for the pair $\gamma^T_{T\setminus\{i\}}$ and $\gamma^{T\cup\{j\}}_{T\cup\{j\}\setminus\{i\}}$. We have also seen that it is true for $c_\varnothing$ and $\gamma^{T\cup\{j\}}_{\{j\}}$. Therefore, we have that it also holds for $c^{T\setminus\{i\}}$ and c^j. Moreover, $c^j\in\Gamma^{T\cup\{j\}}$. By T-dummy Monotonicity, we must have $y_i(T,c^{T\setminus\{i\}},x^T)\geq y_i(T\cup\{j\},c^j,x)$.

Once again using Technological Linearity, the assumption that $y(T,\cdot,\cdot)=\varphi(T,\cdot,\cdot)$ and our selected x, we get that $y_i(T,c^{T\setminus\{i\}},x^T)=2*1+0-1=1$ and $y_i(T\cup\{j\},c^j,x)=2+0-\alpha^1(T\cup\{j\})$.

We obtain

$$\alpha^1(T\cup\{j\})\geq 1$$

Together with (12), this gives that $\alpha^1(T\cup\{j\})=1$.

Step 3: Show that $y(T,\cdot,\cdot)=\varphi(T,\cdot,\cdot)$ for any $T\in\mathcal{N}$

We have shown that $y(T,\cdot,\cdot)=\varphi(T,\cdot,\cdot)$ for all $T\in\mathcal{N}$ such that $|T|=k$, with $1<k<n$ implies that $y(T',\cdot,\cdot)=\varphi(T',\cdot,\cdot)$ for all $T'\in\mathcal{N}$ such that $|T'|=k+1$.

As previously discussed, we have that $y(T,\cdot,\cdot)=\varphi(T,\cdot,\cdot)$ for all $T\in\mathcal{N}$ such that $|T|=2$. Thus, $y(T,\cdot,\cdot)=\varphi(T,\cdot,\cdot)$ for any $T\in\mathcal{N}$ and $y=\varphi$. ■

5 Discussions and extensions

One can weaken the Technological Dummy property by slightly modifying the framework. Suppose that there exists a freely available but poor technology that gives an average cost of $\bar{c}$. The (average) cost vectors are now defined on all subsets of N. In particular, $c(\varnothing) = \bar{c}$. We can now define a weaker technological dummy property by saying that if $c(S \cup \{i\}) = c(S)$ for all $S \subseteq N \backslash \{i\}$, then $y_i(c, x) = \bar{c}x_i$. On this framework, even the weaker Technological Dummy property leaves little latitude. The new degree of freedom consists in how we allocate costs when all agents have average costs of 0, but where $\bar{c} > 0$.[2] It seems extremely natural to impose in those cases that all agents pay 0, which is prescribed by the rule φ.

However using this weaker Technological Dummy property on this framework, Theorem 1, that uses the Stand-alone property, still holds, but Theorem 2, that uses Technological Monotonicity, does not. In the case of Theorem 3, using T-dummy Monotonicity, it follows only if the set of admissible agents is infinite.

The model presented here is obviously simplified, as it imposes constant returns to scale, but shows the importance of the so-called technological effect and how the concept of the Technological Dummy property restricts the set of potential rules. On more general models, we would like to be able to separate the technological and the scale effects, and use appropriate methods on both. Work as yet to be done on this subject, but one intuitive way to separate the two effects would be to estimate the scale effect by evaluating costs at a constant technology, and to estimate technological effect on the function defined by the difference between the real cost function and the one used to isolate the scale effect. A classic cost sharing method could be used to determine the shares on the scale gains, while φ or other methods built with the technological effect in mind could be used to determine the shares on the technological gains.

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[2] Formally, let $\mathcal{N}' = \mathcal{N} \cup \{\varnothing\}$ and $\Gamma' = \left\{c \in \mathbb{R}_+^{\mathcal{N}'} \mid S \subseteq T \Rightarrow c(S) \geq c(T)\right\}$. Define, for $\varnothing \neq T \subset N$, γ'_T such that $\gamma'_T(S) = 0$ if $S \cup T = N$ and 1 otherwise. Define γ'_N such that $\gamma'_N(S) = 1$ for all S and $\gamma'_\varnothing(S) = 0$ for all $S \in \mathcal{N}$ and $\gamma'_\varnothing(\varnothing) = 1$. Define $\Gamma'_0 = \{\gamma'_S\}_{S \in \mathcal{N}'}$.
Γ'_0 is a base of Γ', and on all γ'_T such that $T \neq \varnothing$, the weak and the strong Technological Dummy property imply the same thing, and results are the same as in previous sections. It remains $\gamma'_\varnothing$, for which the weak Technological Dummy property does not say anything.

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6 Appendix

6.1 Properties of φ

Lemma 3 *In fixed-population problems, φ is a budget balanced rule that satisfies the Technological Dummy property, Technological Linearity, Demand Additivity, Equal Shares for Equivalent Agents, Technological Monotonicity, and the Stand-alone property.*

Proof. Fix $i, j \in N$.

Budget balance:

$$\begin{aligned}
\sum_{k\in N} \varphi_k(c,x) &= \sum_{k\in N} c(\{k\})\,x_k - \sum_{k\in N}\sum_{l\in N\setminus\{k\}} x_l Sh_k(v^l(\cdot,c)) \\
&= \sum_{l\in N} c(\{l\})\,x_l - \sum_{l\in N}\sum_{k\in N\setminus\{l\}} x_l Sh_k(v^l(\cdot,c)) \\
&= \sum_{l\in N} c(\{l\})\,x_l - \sum_{l\in N} x_l(c(\{l\}) - c(N)) \\
&= c(N)\sum_{l\in N} x_l = c(N)x(N)
\end{aligned}$$

Technological Dummy: Suppose that c is such that $c(S\cup\{i\}) = c(S)$ for all $\varnothing \neq S \subseteq N\setminus\{i\}$. For any $k \in N\setminus\{i\}$ and any $S \in N\setminus\{i,k\}$. $v^k(S\cup\{i\}, c) = c(S\cup\{i,k\}) - c(\{k\}) = c(S\cup\{k\}) - c(S) = v^k(S,c)$. We also have that $v^k(\{i\}, c) = c(\{i,k\}) - c(\{k\}) = 0$. Thus, all marginal contributions of agent i are equal to zero and $Sh_i(v^k(\cdot,c)) = 0$ for all $k \in N\setminus\{i\}$. Therefore, $\varphi_i(c,x) = c(\{i\})x_i$.

Technological Linearity: For $c^1, c^2 \in \Gamma$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that $\beta_1 c^1 + \beta_2 c^2 \in \Gamma$, we have that, for any $j \in N$, and any $\varnothing \neq S \subseteq N\setminus\{j\}$

$$\begin{aligned}
v^j(S, \beta_1 c^1 + \beta_2 c^2) &= \beta_1 c^1(S\cup\{j\}) + \beta_2 c^2(S\cup\{j\}) - \beta_1 c^1(S) - \beta_2 c^2(S) \\
&= \beta_1 v^j(S,c^1) + \beta_2 v^j(S,c^2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_i(\beta_1 c^1 + \beta_2 c^2, x) &= \beta_1 c^1(\{i\}) + \beta_2 c^2(\{i\}) + \sum_{j\in N\setminus\{i\}} x_j Sh_i(v^j(\cdot, \beta_1 c^1 + \beta_2 c^2)) \\
&= \beta_1 c^1(\{i\}) + \beta_2 c^2(\{i\}) + \sum_{j\in N\setminus\{i\}} x_j \left[\beta_1 Sh_i(v^j(S,c^1)) + \beta_2 Sh_i(v^j(S,c^2))\right] \\
&= \beta_1 \varphi_i(c^1,x) + \beta_2 \varphi_i(c^2,x)
\end{aligned}$$

Demand Additivity. Follows directly from the fact that demands enter φ in a linear fashion.

Equal Shares for Equivalent Agents: Suppose that we have c such that $c(S\cup\{i\}) = c(S\cup\{j\})$ for all $S \subseteq N\setminus\{i,j\}$ and x such that $x_i = x_j$. Clearly, the first terms in $\varphi_i(c,x)$ and $\varphi_j(c,x)$ are equal. For any $S \subseteq N\setminus\{i,j\}$ and any $k \in N\setminus\{i,j\}$

$$v^k(S\cup\{i\},c) = c(S\cup\{i,k\}) - c(\{k\}) = c(S\cup\{j,k\}) - c(\{k\}) = v^k(S\cup\{j\},c)$$

Therefore, $Sh_i(v^k(\cdot,c)) = Sh_j(v^k(\cdot,c)$ for any $k \in N\setminus\{i,j\}$. For any $S \subseteq N\setminus\{i,j\}$, we have

$$\begin{aligned} v^i(S\cup\{j\},c) - v^i(S,c) &= c(S\cup\{i,j\}) - c(\{i\}) - c(S\cup\{i\}) + c(\{i\}) \\ &= c(S\cup\{i,j\}) - c(S\cup\{i\}) \\ v^j(S\cup\{i\},c) - v^j(S,c) &= c(S\cup\{i,j\}) - c(\{j\}) - c(S\cup\{j\}) + c(\{j\}) \\ &= c(S\cup\{i,j\}) - c(S\cup\{j\}) \end{aligned}$$

We thus have that $v^i(S\cup\{j\},c) - v^i(S,c) = v^j(S\cup\{i\},c) - v^j(S,c)$ for any $S \subseteq N\setminus\{i,j\}$, which implies that $Sh_j(v^i(\cdot,c)) = Sh_i(v^j(\cdot,c))$. Therefore, $\varphi_i(c,x) = \varphi_j(c,x)$.

Technological Monotonicity: Fix $x \in \mathbb{R}^n_+$ and $S \in \mathcal{N}$ such that $i \in S$. Take $c, c' \in \Gamma$ such that $c(S) \geq c'(S)$ and $c(T) = c'(T)$ for all $T \in \mathcal{N}\setminus S$. For any $k \in N\setminus\{i\}$, $v^k(S\setminus\{k\},c) = c(S) - c(\{k\}) \geq c'(S) - c(\{k\}) = v^k(S\setminus\{k\},c')$, while $v^k(T,c) = v^k(T,c')$ for all $T \neq S\setminus\{k\}$. By the properties of the Shapley value, $Sh_i(v^k(\cdot,c)) \geq Sh_i(v^k(\cdot,c'))$. Then, $\varphi_i(c,x) \geq \varphi_i(c',x)$.

Stand-alone property: Follows directly from the fact that $Sh_i(v^j,c) \leq 0$ for all $i,j \in N$ and all $c \in \Gamma$, and from the fact that $x \in \mathbb{R}^n_+$ ■

Lemma 4 *In variable population problems, φ is a budget balanced rule that satisfies the Technological Dummy property, Technological Linearity, Demand Additivity, Equal Shares for Equivalent Agents, Technological Monotonicity, the Stand-alone property and T-dummy Monotonicity.*

Proof. Budget balanced, the Technological Dummy property, Technological Linearity, Demand Additivity, Equal Shares for Equivalent Agents, Technological Monotonicity and the Stand-alone property follow from Lemma 3. It remains to look at T-dummy Monotonicity.

Fix $T \in \mathcal{N}\setminus N$, $i \in T$, $j \in N\setminus T$, $x \in \mathbb{R}^{|T|+1}_+$ and $c \in \Gamma^T$ and $c' \in \Gamma^{T\cup\{j\}}$ such that $c'(S) = c(S)$ and $c'(S) = c'(S\cup\{j\})$ for all $S \in \mathcal{T}$.

For any $k \in T$ and $\varnothing \neq S \subseteq T\setminus\{k\}$, $v^k_{T\cup\{j\}}(S,c') = c'(S\cup\{k\}) - c'(\{k\}) = c(S\cup\{k\}) - c(\{k\}) = v^k_T(S,c)$ and $v^k_{T\cup\{j\}}(S\cup\{j\},c') = c'(S\cup\{j,k\}) - c'(\{k\}) = c'(S\cup\{k\}) - c'(\{k\}) = v^k_{T\cup\{j\}}(S,c)$. In addition, $v^k_{T\cup\{j\}}(\{j\},c') = c'(\{j,k\}) - c'(\{k\}) = 0$. By the properties of the Shapley value, $Sh_j(v^k_{T\cup\{j\}}(\cdot,c')) = 0$ and $Sh_i(v^k_{T\cup\{j\}}(\cdot,c')) = Sh_i(v^k_T(\cdot,c))$. Since $Sh_i(v^j_{T\cup\{j\}}(\cdot,c')) \leq 0$, we have $\varphi_i(T\cup\{j\},c',x) \leq \varphi_i(T,c,x^T)$. ■

6.2 Independence of axioms

Define $C(S,c,x) = c(S)x(S)$, the function that assigns the total cost to each coalition. Define $y^1(c,x) = Sh(C(\cdot,c,x))$ as the Shapley value on the stand-alone game C. One can verify that y^1 satisfies Budget-balance, Technological

Linearity, Demand Additivity, Equal Shares for Equivalent Agents, Technological Monotonicity and the Stand-alone property. By example 2, it fails the Technological Dummy property. The adaptation to variable-population problems, $y^1(T,c,x)$, satisfies T-dummy monotonicity.

For all $i \in N$, define $y_i^2(c,x) = c(\{i\})x_i + \frac{\sum_{\varnothing \neq S \subseteq N\setminus\{i\}} c(S\cup\{i\}) - c(S)}{\sum_{j\in N}\sum_{\varnothing \neq S \subseteq N\setminus\{j\}} c(S\cup\{j\}) - c(S)} \left[c(N)x(N) - \sum_{j\in N} c(\{j\})x_j\right]$ if $\sum_{j\in N}\sum_{\varnothing \neq S \subseteq N\setminus\{j\}} c(S\cup\{j\}) - c(S) \neq 0$ and $y_i^2(c,x) = c(\{i\})x_i$ otherwise. One can verify that y^1 satisfies Budget-balance, the Technological Dummy property, Demand Additivity, Equal Shares for Equivalent Agents, Technological Monotonicity and the Stand-alone property, but fails Technological Linearity. The adaptation to variable-population problems, $y^2(T,c,x)$, satisfies T-dummy monotonicity.

Let $D(c) = \{i \in N \mid i \text{ is a technological dummy in } c\}$. For all $i \in N$, define y^3 such that, for all $i \in N$

$$y_i^3(c,x) = \begin{cases} c(\{i\})x_i \text{ if } i \ \in D(c) \\ c(N)x_i + \frac{x_i}{x(N\setminus D(c))}\left[c(N)x(N) - \sum_{j\in D(c)} c(\{j\})x_j\right] \text{ if } i \notin D(c) \text{ and } x(N\setminus D(c)) \neq 0 \\ c(N)x_i + \frac{1}{n-|D(c)|}\left[c(N)x(N) - \sum_{j\in D(c)} c(\{j\})x_j\right] \text{ if } i \notin D(c) \text{ and } x(N\setminus D(c)) = 0 \end{cases}$$

One can verify that y^3 satisfies Budget-balance, the Technological Dummy property, Technological Linearity, Equal Shares for Equivalent Agents, Technological Monotonicity and the Stand-alone property, but fails Demand Additivity. The adaptation to variable-population problems, $y^3(T,c,x)$, satisfies T-dummy monotonicity.

For all $j \in N$ and all $T \in \mathcal{N}$, define $v^{j,T}(\cdot,c)$ as follows:

$$v^{i,T}(S,c) = c(\{S \cup i\}) - c(\{i\}) \text{ for each } S \subseteq T\setminus\{i\}$$

Fix $S \in \mathcal{N}$ and define y^4 such that:

$$y_i^4(c,x) = \begin{cases} c(\{i\})x_i + \sum_{j\in S\setminus\{i\}} x_j Sh_i(v^{j,S}(\cdot,c)) \text{ if } i \in S \\ c(\{i\})x_i + \sum_{j\in N\setminus(S\cup\{i\})} x_j Sh_i(v^{j,N\setminus S}(\cdot,c)) \text{ if } i \in N\setminus S \end{cases}$$

One can verify that y^4 satisfies Budget-balance, the Technological Dummy property, Technological Linearity, Demand Additivity, Technological Monotonicity and the Stand-alone property, but fails Equal Shares for Equivalent Agents. The adaptation to variable-population problems, $y^4(T,c,x)$, satisfies T-dummy monotonicity.

For a problem (c,x), define $\bar{c} = \sum_{S\in\mathcal{N}}(-1)^{|S|+1}c(S)$. Then, define $c' \in \mathbb{R}^{2^n}$ such that $c'(\varnothing) = \bar{c}$ and $c'(S) = c(S)$ for all $S \in \mathcal{N}$. Then, for all $i \in N$, $y_i^5 = \bar{c}x_i + Sh_i(c')x(N)$.

One can verify that y^5 satisfies Budget-balance, the Technological Dummy property, Technological Linearity, Demand Additivity and Equal Shares for Equivalent Agents, but fails Technological Monotonicity and the Stand-alone property. It corresponds to the case where $\alpha^{|T|} = \frac{n-|T|-1}{n-|T|}$. The adaptation to variable-population problems, $y^5(T,c,x)$, fails T-dummy monotonicity.

The following table, where "+" signifies that the property is satisfied, and "-" that it is not, summarizes the results

	φ	y^1	y^2	y^3	y^4	y^5
Technological Dummy property	+	-	+	+	+	+
Technological Linearity	+	+	-	+	+	+
Demand Linearity	+	+	+	-	+	+
Equal Shares for Equivalent Agents	+	+	+	+	-	+
Stand-alone property	+	+	+	+	+	-
Technological Monotonicity	+	+	+	+	+	-
T-dummy Monotonicity	+	+	+	+	+	-