

Between Neighboring Strict Ordinal Games

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1 Introduction and Background

The 2×2 games with strict ordinal preferences for the two players have been given a uniform connecting structure by Robinson and Goforth [5]. The purpose of this paper is to consider what happens between these connections. There are two motivations: understanding differences between adjacent games and accommodating some games with ties.

This new structure on strict ordinal games is best understood mathematically in terms of neighborly transpositions of the symmetric group on four elements, S_4 . A transposition (s, t) is the permutation that swaps s and t , leaving the other numbers fixed. Neighborly transpositions have the form $(s, s + 1)$, swapping adjacent numbers. The neighborly transpositions $(1, 2)$, $(2, 3)$, and $(3, 4)$ generate all 24 permutations of S_4 .

Strict ordinal 2×2 games can be denoted as the direct product $S_4 \times S_4$ (which has $24 \cdot 24 = 576$ elements), which is equivalent to the standard matrix representation of ordered pairs. One can also use “order graphs” of four points and four edges, which partition the 576 matrices into equivalence classes. Equivalent games can also be identified from the matrix representations, although there is not unanimity on how to do this. But, to quote from the author’s review of [5] in *Mathematical Reviews* [4]:

One departure from earlier analysis is the number of such games: by acknowledging less symmetry than A. Rapoport and M. J. Guyer, the current authors arrive at 144 games rather than the established 78. To the reviewer, this distinction is not so important, comparable to whether one counts 21 or 36 possibilities when rolling two six-sided dice—correct analysis can follow from either convention.

The major contribution of Robinson and Goforths system is connecting the games via “neighborly transpositions” from the study of symmetric groups.

More specifically, Robinson and Goforth use six operations to connect the 144 games, sets of the neighborly transpositions for each player. That is, the row player’s preferences can be altered by R_{12} , R_{23} , or R_{34} in the first coordinate of $S_4 \times S_4$, and the column player’s preferences can be altered by C_{12} , C_{23} ,

or C_{34} in the second coordinate. (They also use composed operations such as $S_{12} = R_{12} \circ C_{12}$ which move a symmetric game to another symmetric game; note that $R_{12} \circ C_{12} = C_{12} \circ R_{12}$.)

The resulting graph of 144 vertices and $\frac{144 \cdot 6}{2} = 432$ edges (each vertex has six edges, and each edge connects two vertices) is shown to have genus 37, meaning that it can be embedded on the two-dimensional surface of a torus that has 37 holes. While this may not be very helpful in visualizing the entire structure of the 144 games, aspects of this topological structure will be used in §3.

2 Looking Between to Explain Changes

Previous groupings of strict ordinal 2×2 games were done by somewhat *ad hoc* means, such as by number of Nash equilibria. The Robinson and Goforth structure is preferable for its mathematical simplicity, consistency, and completeness. Because of the completeness, though, it necessarily makes adjacent games that have very different characteristics. Looking between such games allows one to better understand the changes between such games.

To define the notion of “between games,” we relax the conventions of ordinal preference. The operation S_{12} , for instance, swaps 1 and 2 in the preferences for both the row and column players. Think of this as a continuous transition $1 + \alpha$ and $2 - \alpha$ as α ranges from 0 to 1, rather than the discrete transposition. In terms of ordinal preference,

$$(1 + \alpha, 2 - \alpha) \sim \begin{cases} (1, 2) & \text{if } \alpha < \frac{1}{2}, \\ (1, 1) & \text{if } \alpha = \frac{1}{2}, \\ (2, 1) & \text{if } \alpha > \frac{1}{2}, \end{cases}$$

where \sim denotes ordinal equivalence. However, it is helpful to use $(1.5, 1.5)$ for the case $\alpha = \frac{1}{2}$. This case is called a “half-swap” in the working paper of Robinson, Goforth, and Cargill [6].

This continuous transition is perhaps easiest to understand in the setting of order graphs, where the point $(1, 4)$, for instance, moves smoothly through $(1.5, 4)$ on its way to $(2, 4)$ under the S_{12} (or R_{12}) operation.

Consider the well-known games Prisoner’s Dilemma and Chicken, which are connected by the symmetry-preserving S_{12} operation. Prisoner’s Dilemma famously has one Nash equilibrium, which is not Pareto optimal, while Chicken has two Nash equilibria. Looking at the game between them is instructive. We follow the convention of order graphs in [5]: single lines connect the row player choices, dotted lines the column player choices, and white vertices denote Nash equilibria.

In Figure 1, the point $(1, 4)$ transitions through $(1.5, 4)$ before reaching $(2, 4)$. Notice the the y -coordinate 4 does not change, as it is not effected by S_{12} . Similarly, the point $(4, 1)$ transitions through $(4, 1.5)$ before reaching $(4, 2)$. The

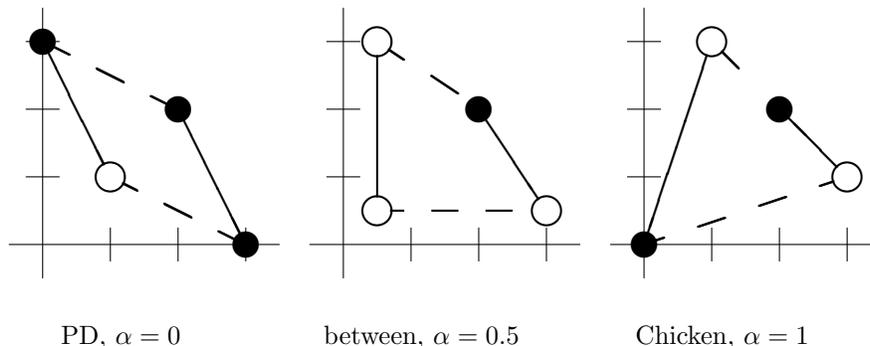


Figure 1: S_{12} connecting Prisoner's Dilemma and Chicken.

point $(2, 2)$ transitions through $(1.5, 1.5)$ before reaching $(1, 1)$, while the point $(3, 3)$ is unchanged by S_{12} .

The between game helps one understand the change between the very different Nash equilibria characteristics of Prisoner's Dilemma and Chicken. The point $(2 - \alpha, 2 - \alpha)$ is a Nash equilibrium for $\alpha \leq \frac{1}{2}$, a range that includes Prisoner's dilemma and the between game. The points $(1 + \alpha, 4)$ and $(4, 1 + \alpha)$ are Nash equilibria for $\alpha \geq \frac{1}{2}$, a range that includes Chicken and the between game. In this between game, occurring at $\alpha = \frac{1}{2}$, there are three Nash equilibria. At this value of α , both players are indifferent between certain choices, corresponding to horizontal and vertical line segments in the order graph. Thinking of α moving from 0 to 1, the between game includes the Pareto non-optimal Nash equilibrium "on its way out" and the two Nash equilibria from Chicken "just starting."

This analysis of between game can be done for any pair of adjacent strict ordinal 2×2 games, either swapping values for both players simultaneously (as with S_{12} here) or swapping values for a single player (used in the next section). Whenever there is a change in the number or relative position of Nash equilibria, the between game with its indifference can help elucidate the transition.

3 Looking Between to Accommodate Ties

In situations modeled by 2×2 games, it is not always the case that both players have a strict preference among all four possible outcomes. A player could be indifferent between multiple options. Although the structure described by Robinson and Goforth in [5] connects the 144 games with strict ordinal preferences, it can be expanded by various notions of between games to include several 2×2 games that include ties.

Recall that the 144 vertex, 432 edge graph can be embedded on a genus

37 torus—this is an example of a 2-manifold, meaning that locally it can be considered as the two dimensional plane. Using a technique from map-coloring problems in graph theory, consider each of the 144 games not as a vertex but rather a face. (Visually, consider expanding each vertex “dot” until it presses against its neighbors, leaving no gaps.) The six adjacent games are now bordering faces; each game corresponds to a hexagon. The edge between two bordering faces indicates the appropriate row or column swap operation. These edges meet in various vertices (where three or more faces come together). We will incorporate several 2×2 games with ties into Robinson and Goforth’s structure by assigning such games to these edges and vertices.

We examine in detail the games connected to Prisoner’s Dilemma by the R_{12} and C_{12} operators, now considered separately. Between PD and the asymmetric game $R_{12}(\text{PD})$ is a game where two points in the order graph have x -coordinate 1.5 (note the R_{12} has no effect on the y -coordinates, the column player’s preferences). These three games are shown in the bottom row of Figure 2; the between game is labeled “edge: 3 vs. 4” to highlight that the row player has three preferences among outcomes of the 2×2 game while the column player has four. (More specifically, Fraser and Kilgour [1] would label this a 1123 vs. 1234 game; we use 1.5, 1.5, 3, 4 rather than 1, 1, 2, 3.) This between game (with two Nash equilibria) is assigned to the edge between PD and $R_{12}(\text{PD})$.

The entirety of Figure 2 calls for extensive explanation. There is an analogous 4 vs. 3 game corresponding to the edge between the PD and $C_{12}(\text{PD})$ faces, shown in the leftmost column. As these two operators commute, the fourth face indicated in the figure is $C_{12}(R_{12}(\text{PD})) = R_{12}(C_{12}(\text{PD})) = \text{Chicken}$. The four edges between the adjacent pairs of these faces correspond to games where one player has four distinct preferences, the other three. These four edges meet at a vertex; the game assigned to the vertex is the between game from §2 where both players are half-way between swapping 1 and 2, a 3 vs. 3 game.

For the nine games shown in Figure 2, each row indicates three stages of swapping 1 and 2 for the row player: initial game, half-swap, and the finished swap. Each column indicates the same three stages for the column player. Although each game is shown as an order graph, they correspond to different objects in this expanded structure based on Robinson and Goforth. The games with strict preferences for both players (here the corners) correspond to faces of the torus map. The games where one player has four preferences, the other three (here the middle of the outside edges) correspond to edges of the torus map. Finally, the game where both players have three preferences (here in the middle) corresponds to the vertex where the four edges and the four faces meet. Notice that the three games of Figure 1 are contained in Figure 2, specifically the bottom left, middle, and upper right games.

If these various between games are added throughout the 144 face torus map, how many games with ties are incorporated into this expanded structure? Games with ties were enumerated by Fraser and Kilgour [1] and revised by Robinson, Goforth, and Cargill [6] to remove the “Rapoport and Guyer reflections” (which leads to 144 count for strict preference games rather than 78). We will see that all but some games involving a player having only one or two

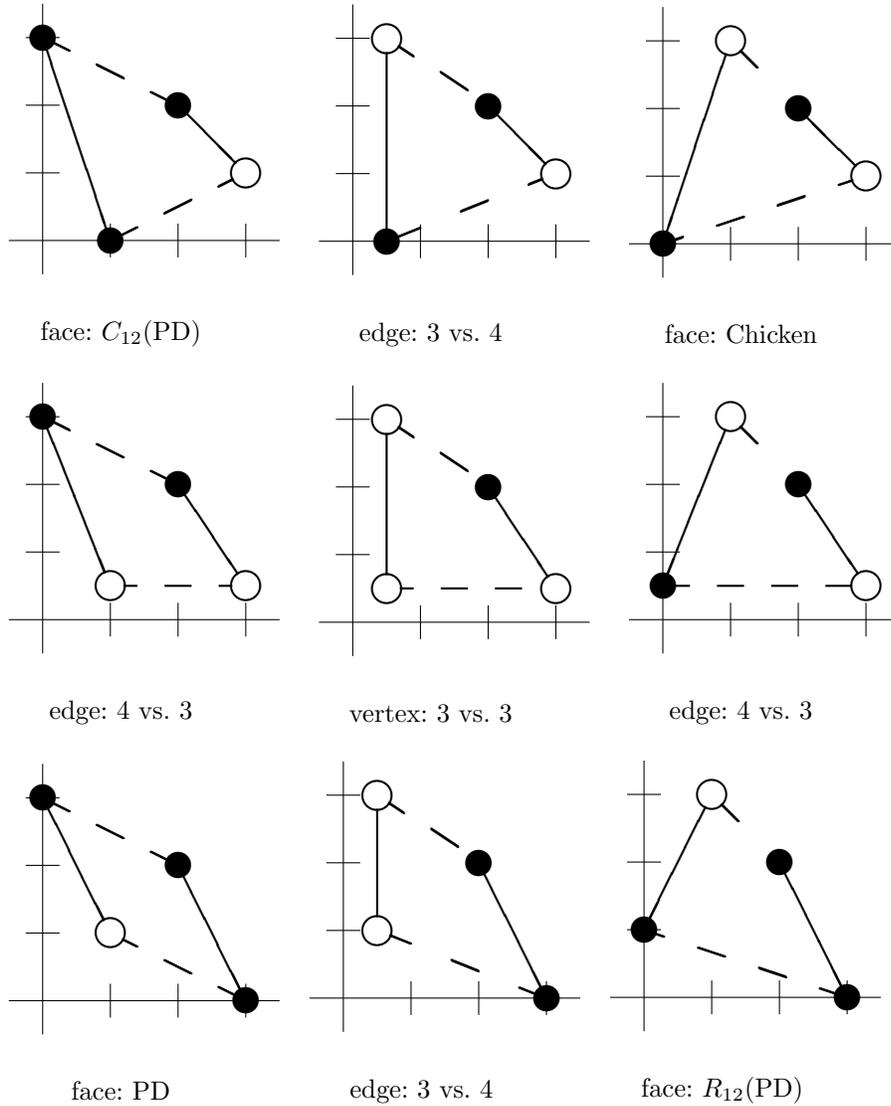


Figure 2: R_{12} (horizontal) and C_{12} (vertical) applied to Prisoner's Dilemma.

distinct preferences are accounted for in this expanded structure.

Fraser and Kilgour count 432 games where one player has four preferences and the other three. These are the edges in our expanded structure. Each game face is bordered by six edges, and each edge is shared by two faces, accounting

for all $\frac{144 \cdot 6}{2} = 432$ such games.

To count vertices of the expanded structure, one needs to consider pairs of the six operations, of which there are $\binom{6}{2} = 15$. The $3 \cdot 3 = 9$ pairs with one column and one row operator are analogous to Figure 2, with a game where both players have three preferences corresponding to the vertex. Each of the 144 face games is associated with 9 of these vertex games, and each such vertex game corresponds to 4 strict preference games (such as the corners in Figure 2), giving $\frac{144 \cdot 9}{4} = 324$ of these 3 vs. 3 games, which matches the revised Fraser and Kilgour count.

The two pairs (R_{12}, R_{34}) and (C_{12}, C_{34}) also each consist of commuting operators and produce a grid of nine games, but both half-swaps reduce the preferences of the same player. Here, the games corresponding to the vertices have two preferences for one player and four for the other (specifically, Fraser and Kilgour types 1122 and 1234). Similar to the enumeration above, this gives $\frac{144 \cdot 2}{4} = 72$ such games, matching the earlier count.

The remaining four pairs of operators do not consist of commuting operators. Rather they are connected by the braid relation $R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}$, for example. The geometric interpretation of half-swaps does not provide an understanding of what game might occur at intersections of these lines.

In conclusion, this expanded structure based on Robinson and Goforth accommodates 972 games: the 144 strict preference 2×2 games (now faces on a torus), all 432 games of the forms 4 vs. 3 and 3 vs. 4 preferences (edges), the 324 games with 3 vs. 3 preferences (some of the vertices), and all 72 games of the forms 2 vs. 4 and 4 vs. 2 preferences where the 2 indicates two low and two high preferences (other vertices). (Compare Robinson, Goforth, and Cargill [6] which represents 900 games similarly, excluding the final 72 games mentioned here.)

Goforth and Robinson [2, to appear] have done a similar analysis on the 12 symmetric strict ordinal 2×2 games using half-swaps of operators such as S_{12} discussed in §2. In the “winged octahedron” of [5], the 12 games correspond to triangular faces, the 18 edges represent symmetric games with three preferences for each player, and the seven vertices represent symmetric games with two preferences for each player. (In the octahedron model, the two “wing tips” actually correspond to the same point.) Similar work was done independently in the undergraduate honors thesis of Sarah Heilig, SPC ’11 [3].

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